METRICS ON FRACTALS AND SUB-GAUSSIAN HEAT KERNEL ESTIMATES

QINGSONG GU, KA-SING LAU, HUA QIU, AND HUO-JUN RUAN

Abstract. It is well-known that for a Brownian motion, if we change the medium to be inhomogeneous by a measure μ, then the new motion (time changed process) will diffuse according to a different metric $D(\cdot,\cdot)$. In [22], Kigami initiated a general scheme to construct such metrics through some self-similar weight functions g on the symbolic space.

In order to provide concrete models to Kigami's theoretical construction, in this paper, we give a thorough study of his metric on two classes of fractals of primary importance: the nested fractals and the generalized Sierpinski carpets; we assume further that the weight functions $g := g_{\boldsymbol{a}}$ are generated by "symmetric" weights $\boldsymbol{a}$. Let $\mathcal{M}$ be the domain of $\boldsymbol{a}$ such that $D_{g_{\boldsymbol{a}}}$ defines a metric, and let S be the boundary of $\mathcal{M}$. One of our main results is that the metrics from $g_{\boldsymbol{a}}$ satisfy the metric chain condition (MCC) if and only if $\boldsymbol{a} \in S$. To determine $\mathcal{M}$ and S, we provide a recursive weight transfer construction on the nested fractals, and a basic symmetric argument on the Sierpinski carpet. As an application, we use the MCC to obtain the lower estimate of the sub-Gaussian heat kernel. This together with the upper estimate in [22] allows us to have a concrete class of metrics for time change, and the two sided sub-Gaussian heat kernel estimate on the fundamental fractals.

Contents

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1. Introduction

Metric spaces play a prominent role in various fields in mathematics. The analysis on metric spaces together with measures (metric measure spaces) emerged as an independent research field since the 90's. The spaces have no *a priori* smooth structure, but one is able to recover the infinitesimal concepts such as gradient, Laplacian, Dirichlet form, and curvature as in Euclidean function theory, geometric analysis and stochastic analysis [10, 16, 17, 32]. In the analysis on fractals, a wealth of exotic examples and different metrics have emerged due to self-similarity. This also provides a fertile background for the theory of metric measure spaces (see e.g. [20, 23, 13]).

In the study of Brownian motion on the Sierpinski gasket (SG), Barlow and Perkins [8] first established the Li-Yau type sub-Gaussian estimate of the transition density function

$$p_t(x,y) \asymp \frac{1}{V\left(x, t^{1/\beta}\right)} \exp\left(-c\left(\frac{d^{\beta}(x,y)}{t}\right)^{1/(\beta-1)}\right), \tag{1.1}$$

with $d(\cdot,\cdot)$ as the Euclidian metric, $V(x,t^{1/\beta}) = \mu\{z \in \mathrm{SG} : d(x,z) \leq t^{1/\beta}\} \asymp t^{\alpha/\beta}$, where μ is the canonical α-dimensional Hausdorff measure on the SG. The Sierpinski gasket has energy renormalization factor $\rho = 3/5$, and walk dimension $\beta = \log 5/\log 2$ [8]. This was extended by Lindstrøm [29] by showing that the Brownian motion exists on a class of self-similar sets called *nested fractals*, and the transition density of the Brownian motion on the nested fractal (with a technical path assumption) was shown to enjoy the two sided sub-Gaussian estimate by Kumagai [26]. A path breaking extension was proved on the Sierpinski carpet (SC) by Barlow and Bass in their seminal papers [2, 4], with $\beta = \frac{\log 8\rho^{-1}}{\log 3}$ (only approximate value of ρ is available).

If we change the medium to be inhomogeneous by a measure μ, then the new motion will have the same paths, but different rate of diffusion, and is associated with a different metric $D(x,y)$; we call it a *time change* of the process. One of the main issues is to maintain the sub-Gaussian estimate (1.1) with the new metric $D(x,y)$. The time change for self-similar measures μ on *p.c.f.* sets that admit harmonic structures and local regular Dirichlet forms and on the SC were first studied by Barlow and Kumagai [6], and they showed that the time change is possible if $\rho_i\mu_i < 1$ for all $1 \leq i \leq N$, where μ_i's are the probability weights of μ.

In [21, 22, 23], Kigami launched a detail study of the time change problem in full generality based on the Dirichlet forms and the resistance metrics. He set up a general scheme to construct new metrics $D(x,y)$ on fractals. From the point of view of local regular Dirichlet forms and the associated Hunt processes, the metric $D(x,y)$ is closely connected with the resistance metric $R(x,y)$ on the Dirichlet space described by the Einstein relation (see [33]).

$$R(x,y)V_D(x,D(x,y)) \asymp D(x,y)^{\beta}.$$

In this paper, we adopt the same setup as [22, 23] to construct metrics by weight functions on the iterated function system (IFS) of fractals. In Kigami's study, there were few concrete examples or discussion on the class of admissible weight functions. For this reason, we will restrict our consideration to the two most basic classes of fractals: the nested fractals and generalized Sierpinski carpets (GSC) (see the definition in Section 2). When we do not need to distinguish the two classes, we will just call them **elementary fractals** for convenience. We will consider the metrics arising from the class of "symmetric self-similar weights" (Definition 1.2). The techniques used throughout the paper depend very strongly on the group of symmetries of the underlying set, which is quite different from the previous investigations. The study leads to new results on the class of admissible metrics for time change, and sharpens the sub-Gaussian heat kernel estimate.

Let $\{F_i\}_{i=1}^N$ denote the associated IFS of an elementary fractal K. For $n \geq 1$, let $\Sigma^n = \{1, \ldots, N\}^n$ be the collection of *words* with length n (by convention, $\Sigma^0 = \{\emptyset\}$). For $w = w_1 \cdots w_n \in \Sigma^n$, we write $K_w = F_w(K) := F_{w_1} \circ \cdots \circ F_{w_n}(K)$, and call it an *n-cell* of K. Denote by $\Sigma^* = \bigcup_{n \geq 0} \Sigma^n$ the collection of all finite words, and by $|w|$ the *length* of w for each $w \in \Sigma^*$. A finite sequence of words $(w(1), \ldots, w(m))$ in Σ^* (or equivalently, cells $(K_{w(1)}, \ldots, K_{w(m)})$ in K) is called a *chain* if $K_{w(i)} \cap K_{w(i+1)} \neq \emptyset$ for $1 \leq i \leq m-1$; we use $|\gamma| = m$ to denote the length of the chain. A chain $(w(1), \ldots, w(m))$ is said to connect x and y if $x \in K_{w(1)}$ and $y \in K_{w(m)}$. A chain is called *simple* if $K_{w(i)} \cap K_{w(j)} \neq \emptyset$ if and only if $|i - j| \leq 1$.

Definition 1.1 ([24, 25]). *We call $g : \Sigma^* \to (0, 1]$ a weight function if it satisfies:*
(i) $g(\emptyset) = 1$, $g(wj) \leq g(w)$ if $w \in \Sigma^*$ and $j \in \{1, \ldots, N\}$;
(ii) $\lim_{n\to\infty} \sup_{w \in \Sigma^n} g(w) = 0$.

We define the total weight of a chain $\gamma = (w(1), \ldots, w(m))$ by $g(\gamma) = \sum_{i=1}^m g(w(i))$, and for any $x, y \in K$,

$$D_g(x, y) = \inf\Big\{g(\gamma) : \gamma \text{ is a chain connecting } x \text{ and } y\Big\}. \tag{1.2}$$

It is easy to see that $D_g(\cdot, \cdot)$ is finite ($D_g \leq g(\emptyset) = 1$), symmetric, nonnegative, $D_g(x, x) = 0$ for all $x \in K$, and satisfies the triangle inequality. However, in general, it may happen that $D_g(x, y) = 0$ for some pairs $x \neq y$ in K so that D_g fails to be a metric.

Let G be the group of symmetries associated with the elementary fractal K (see Section 2). We will focus on the class of weight functions as following.

Definition 1.2. *We call $g : \Sigma^* \to (0, 1]$ a* symmetric self-similar weight function *if g satisfies the following two conditions:*
(*i*). (Self-similarity) *$g(w) = \prod_{i=1}^m g(w_i)$ for $w = w_1 w_2 \cdots w_m \in \Sigma^*$.*
(*ii*). (Symmetry) *For all $\sigma \in G$, $g \circ \sigma = g$.*

We remark that in the above definition (ii), for $\sigma \in G$, σ acts on the cells K_w. Since the cells and the finite words in Σ^* are in $1-1$ corresponding, we can define the procedure of σ on Σ^*. For any $i, j \in \Sigma$, define $i \sim_G j$ if there is a $\sigma \in G$ such that $K_j = \sigma(K_i)$. Let $\Sigma^{\sim_G}$ denote the equivalent classes and $k = \#\Sigma^{\sim_G}$. For example, the Sierpinski gasket and the pentagasket have $k = 1$; the more interesting cases are the Lindstrøm snowflake and the standard Sierpinski carpet with $k = 2$ (Sections 5, 6).

First by self-similarity and reflecting the cells along hyperplanes of symmetry, we prove an interesting dichotomic result.

Theorem 1.3. *Let K be an elementary fractal, and let g be a symmetric self-similar weight function. Then $D_g(\cdot,\cdot)$ is either a metric or identically equal to* 0*.*

Let $\boldsymbol{a} := (a_1, a_2, \dots, a_k) \in (0,1)^k$ be the associated weights of $\{g(i) : 1 \le i \le N\}$. We write $g = g_{\boldsymbol{a}}$ for the weight function associated with respect to $\boldsymbol{a}$. We define

$$\mathcal{M} := \{\boldsymbol{a} \in (0,1)^k : \ D_{g_{\boldsymbol{a}}} \text{ is a metric on } K\},$$

and call it the set of *admissible weights*, and $D_{g_{\boldsymbol{a}}}$ an *admissible metric* (for time change). We have (Propositions 2.7 and 3.5).

Proposition 1.4. *Let K be an elementary fractal, and let $S = \partial\mathcal{M} \cap (0,1)^k$ be the boundary of $\mathcal{M}$. Then $\mathcal{M}$ is closed, and S separates $(0,1)^k$ into two connected components $\mathcal{M}$ and $\mathcal{M}^c$, with $S \subset \mathcal{M}$.*

There is an expression for $\boldsymbol{a} \in \mathcal{M}$, which is convenient to use in the sequel (Theorem 1.8). For $\boldsymbol{a} \in \mathcal{M} \subset (0,1)^k$ and $\lambda \in (0,\infty)$, consider $\boldsymbol{a}(\lambda) = (a_1^\lambda, \dots, a_k^\lambda)$, then $\boldsymbol{a}(1) = \boldsymbol{a}$, and $\lim_{\lambda\to 0} \boldsymbol{a}(\lambda) = (1,\dots,1)$, $\lim_{\lambda\to\infty} \boldsymbol{a}(\lambda) = (0,\dots,0)$. We show that $\boldsymbol{a}(\lambda) \in \mathcal{M}$ for λ small, and $\boldsymbol{a}(\lambda) \in \mathcal{M}^c$ if λ is large (see Section 2 and Figure 1). Hence there is a unique λ_0 such that $\boldsymbol{a}(\lambda_0) \in S$.

Recall that the main purpose to study the admissible metrics D_g is to obtain a two-sided sub-Gaussian heat kernel estimate (1.1) with respect to D_g. For the off-diagonal lower estimate in the sub-Gaussian heat kernel, one requires the metric to satisfy the *metric chain condition* (see e.g. [11, 15]). We also remark that the two-sided sub-Gaussian heat kernel estimate does imply the metric chain condition (see [30, Corollary 1.8]).

Definition 1.5. *A metric space (M,d) is said to satisfy the* metric chain condition *(MCC) if there exists a constant $C > 0$ such that for any two points $x, y \in M$ and for any positive integer n, there exists a sequence $\{x_i\}_{i=0}^n$ of points in M such that $x_0 = x$, $x_n = y$ and*

$$d(x_i, x_{i+1}) \le C\frac{d(x,y)}{n}, \ \textit{for all } i = 0, 1, \dots, n-1.$$

The MCC plays an important role in the lower estimate of the sub-Gaussian heat kernel (see e.g. [12]). The following theorem is one of our main results (Lemma 4.2, Theorem 4.3).

Theorem 1.6. *Let K be an elementary fractal, then an admissible metric $D_{g_{\boldsymbol{a}}}$ satisfies the MCC if and only if $\boldsymbol{a} \in S$.*

It is a challenging task to identify the set of admissible weights $\mathcal{M}$, and there is no results or non-trivial examples in literature. Our next goal is to give a detail study of this for the elementary fractals. For nested fractals, we use a technique of Kumagai [26] to give a constructive algorithm: for each $\boldsymbol{a} \in \mathcal{M}$, there is a recursive relation on the weights of the paths on each level. This allows us to formulate a finite family of "weight transfer matrices" $\mathcal{K}(\boldsymbol{a})$. Let λ_A be the maximal positive eigenvalue of a matrix A, we have (Theorem 5.1)

Theorem 1.7. *For a nested fractal, the set of admissible weights is*

$$\mathcal{M} = \{\boldsymbol{a} = (a_1, \dots, a_k) \in (0,1)^k : \ \lambda_A \geq 1, \ \forall\, A \in \mathcal{K}(\boldsymbol{a})\}$$

and its boundary $S = \{\boldsymbol{a} \in \mathcal{M} : \ \exists\, A \in \mathcal{K}(\boldsymbol{a}) \ \ni \ \lambda_A = 1\}$.

We use the Lindstrøm snowflake (see Section 5 and Figure 3) as an example to illustrate the theorem. For the Sierpinski carpet, it requires a different technique to identify $\mathcal{M}$. We will give a detail consideration of this in Section 6.

We then apply the above results to the time change problem. It is well-known that for a nested fractal K, if we denote by α the Hausdorff dimension, and let $\mathcal{H}^\alpha$ be the normalized α-dimensional Hausdorff measure, then there exists a local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mathcal{H}^\alpha)$ satisfying the self-similar energy identity with a uniform renormalization factor $0 < \rho < 1$, that is

$$\mathcal{E}(u) = \frac{1}{\rho} \sum_{i=1}^{N} \mathcal{E}(u \circ F_i), \qquad \forall u \in \mathcal{F}, \tag{1.3}$$

and the induced process is the Brownian motion. For a GSC in $\mathbb{R}^d$ with $d \leq 2$, we also have $0 < \rho < 1$. It may happen that $\rho \geq 1$ for $d \geq 3$ (see [5, Remarks 5.4]). If we let μ be a self-similar measure on these fractals with weights μ_i, i.e., $\mu = \sum_{i=1}^N \mu_i\, \mu \circ F_i^{-1}$, then in the case that $\mu_i \rho < 1$ for all $1 \leq i \leq N$, the measure defines a new local regular Dirichlet form $(\mathcal{E}, \mathcal{F}')$ in $L^2(K, \mu)$ with the same $\mathcal{E}$, which also induces a diffusive process [6].

We call a self-similar measure μ *symmetric* if $\mu_i = \mu_{\sigma(i)}$ for any $\sigma \in G$ and $i \in \Sigma$. It is known that symmetric self-similar measures are doubling measures under the admissible metrics [22, Theorem 3.4.5] (also see Section 7).

With this setup on elementary fractals, the sub-Gaussian heat kernel estimate of the time change of Brownian motion for symmetric self-similar measures can be stated precisely.

Theorem 1.8. *Let K be an elementary fractal. Let μ be a symmetric self-similar measure, and let $\boldsymbol{a}(\lambda)$ be the curve defined by*

$$\boldsymbol{a}(\lambda) = \Big((\mu_1\rho)^\lambda, (\mu_2\rho)^\lambda, \dots, (\mu_k\rho)^\lambda\Big) \in \mathcal{M}, \qquad \lambda \in (0, \infty). \tag{1.4}$$

Let $\beta = 1/\lambda$, and D_g be the admissible metric defined by $\boldsymbol{a}(\lambda)$. Then the time change of Brownian motion with measure μ has a transition density $p_t(x,y)$ that admits an *upper sub-Gaussian estimate* (UE)

$$p_t(x,y) \le \frac{C}{V(x, t^{1/\beta})} \exp\left(-c\left(\frac{D_g(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right),$$

where $V(x,r) := \mu(\{z :\ D_g(x,z) < r\})$*, and a near diagonal lower estimate* (NLE)*: there exists small* $\eta > 0$ *such that*

$$\frac{C}{V(x, t^{1/\beta})} \le p_t(x,y), \quad \forall\ x, y \in K,\ t > 0,\ D_g(x,y) < \eta t^{1/\beta}.$$

In particular if $\lambda = \lambda_0$ *such that* $\boldsymbol{a}(\lambda_0) \in S$*, then* $p_t(x,y)$ *has the two sided sub-Gaussian estimate as in* (1.1).

The upper estimate and the near diagonal lower estimate (NLE) were proved by Kigami [22, Theorem 3.2.3] for weights under the more general situation. He also showed the off-diagonal lower estimate holds for x, y if there exists a geodesic path between them. Our contribution in Theorem 1.8 is to provide concrete families of admissible metrics with MCC for $\boldsymbol{a} \in S$ so that the off-diagonal lower estimate can be assured. The proof in [22] (see also [14]) are quite involved and lengthy, therefore, we give an outline of their proofs incorporating with our setup. We make use of the fact that these admissible metrics are quasisymmetric to the resistance metric for nested fractal, and to the Euclidean metric for GSC [23, 24], and the classical techniques of capacity estimate and Harnack inequality in [14].

We remark that there are few examples on non-symmetric weight functions in connection to those considered in [22]. We also remark that there is another setup to construct new metrics on self-similar sets which is quite different from the present one; the construction is based on certain "augmented trees", i.e., by adding new edges to the neighboring cells of the trees of the symbolic spaces; the geodesic of these trees are defined by the graph distance. They are hyperbolic graphs (in the sense of Gromov), and there are systematic treatments for such graphs [18, 19, 27, 28]. Also in regard to the quasisymmetry of the SC, Bonk and Merenkov in [9] gave an interesting classification of quasisymmetric self-homeomorphisms for the standard $1/3$-SC and the $1/\ell$-SC.

We organize the paper as follows. In Section 2, we provide some basic definitions and preliminaries of the elementary fractals and the symmetric weight functions. We prove some basic facts of $\mathcal{M}$, $\mathcal{M}^c$ and the boundary $S = \partial\mathcal{M} \cap (0,1)^k$. In Section 3, we verify Theorem 1.3 for D_g to be identically zero in $\mathcal{M}^c$. The main theorem (Theorem 1.6) on the MCC is proved in Section 4. In Section 5, we prove Theorem 1.7 and use it on the Lindstrøm snowflake for illustration. In Section 6, we study $\mathcal{M}$ of the Sierpinski carpet in detail. Finally in Section 7, we combine Theorem 1.6 together with some previous known results to obtain the heat kernel estimates of the time change Brownian motion for symmetric self-similar measures (Theorem 1.8).

2. Preliminaries and admissible metrics $D_g(\cdot,\cdot)$

First we define the class of *nested fractals* introduced by Lindstrøm [29]. Let K be the self-similar set defined by an iterated function system (IFS) $\{F_i\}_{i=1}^N$ of the form $F_i(x) = \varrho O_i x + b_i$, where $N \geq 2$, $0 < \varrho < 1$, and for each $1 \leq i \leq N$, O_i is a $d \times d$ orthogonal matrix and $b_i \in \mathbb{R}^d$. Let P be the set of all fixed points of $\{F_i\}_{i=1}^N$. Call $p \in P$ an *essential fixed point* if there exist distinct $i, j \in \{1, \ldots, N\}$, and $q \in P$ such that $F_i(p) = F_j(q)$, and denote this set by P_0. For any distinct points $x, y \in \mathbb{R}^d$, denote the bisecting hyperplane $H_{x,y} = \{z \in \mathbb{R}^d : |x - z| = |y - z|\}$ and write $R_{x,y}$ the orthogonal reflection with respect to $H_{x,y}$; let G denote the group of reflections for $x, y \in P_0$.

Definition 2.1 (**nested fractals**). *Let $\{F_i\}_{i=1}^N$ and K be as the above. We call K a* nested fractal *if it satisfies the following conditions:*

*(*OSC*) $\{F_i\}_{i=1}^N$ satisfies the open set condition;*
*(*Connectivity*) K is connected;*
*(*Symmetry*) K is invariant under G;*
*(*Nesting*) For any $i, j \in \{1, \ldots, N\}$ with $i \neq j$, $F_i(K) \cap F_j(K) = F_i(P_0) \cap F_j(P_0)$.*

Next we define another class of self-similar sets which are infinitely ramified, called *generalized Sierpinski carpets* (GSC), named and first studied by Barlow and Bass [2, 4]. Let $d \geq 2$, $\ell \geq 3$ be integers, and $H_0 = [0, 1]^d$. Set $\mathcal{Q}$ to be the mesh of closed subcubes of size $1/\ell$. For any $Q \in \mathcal{Q}$, let $F_Q : H_0 \to H_0$ be given by $F_Q(x) = x/\ell + p_Q$ where p_Q is chosen so that $F_Q(H_0) = Q$. Let $\mathcal{Q}' \subseteq \mathcal{Q}$ and let $K := \mathrm{GSC}(d, \ell, \mathcal{Q}')$ be the self-similar set associated with the iterated function system $\{F_Q\}_{Q \in \mathcal{Q}'}$. We renumber the elements in $\{F_Q\}_{Q \in \mathcal{Q}'}$ by $\{F_i\}_{i=1}^N$ with $N = \#\mathcal{Q}'$. Set $H_1 = \bigcup_{Q \in \mathcal{Q}'} F_Q(H_0)$. Let G denote the group of isometries on H_0.

Definition 2.2 (**generalized Sierpinski carpets**). *A set $K = GSC(d, \ell, \mathcal{Q}')$ is called a* generalized Sierpinski carpet *(GSC) if the following conditions are satisfied:*

*(*Symmetry*) H_1 is invariant under G;*
*(*Connectivity*) H_1 is connected;*
*(*Non-diagonality*) For any $x \in H_1$, there exists $r_0 > 0$, such that for all $0 < r < r_0$, $int(H_1 \cap B(x, r))$ is connected;*
*(*Borders included*) The line segment $[0, 1] \times \{0\} \times \cdots \times \{0\}$ is contained in H_1.*

Throughout the paper we always assume that K is either a nested fractal or a GSC. When we do not need to distinguish them, we will just call them *elementary fractals* for convenience.

A weight function g will be assumed to be self-similar and symmetric as in Definition 1.2. For the weight function $a_i = g(i)$, $i \in \Sigma = \{1, 2, \ldots, N\}$, by taking quotient of symmetries, we consider $\boldsymbol{a} \in (0, 1)^k$ where k is the number of elements in the quotient space $\Sigma^{\sim_G}$. For a chain $\gamma = (w(1), \ldots, w(m))$, the weight of a chain $\gamma = (w(1), \ldots, w(m))$ is defined by

$$g(\gamma) = \sum\nolimits_{i=1}^m g(w(i)) \quad \text{with} \quad g(w(i)) = \prod\nolimits_{j=1}^n g(i_j), \;\; w(i) = i_1 \cdots i_n.$$

For the above chain, we also define the *union* of the cells in γ by $\cup\gamma = \bigcup_{i=1}^{m} K_{w(i)}$, and call $(w(i), \dots, w(j))$ a *sub-chain* of γ for any $1 \le i \le j \le m$.

For any two words w and v, if K is a nested fractal, we use $K_w \sim K_v$ to denote $K_w \cap K_v \neq \emptyset$; if K is a GSC in $\mathbb{R}^d$, we use $K_w \sim K_v$ to mean $\dim(F_w(H_0) \cap F_v(H_0)) \ge d-1$, i.e., either $F_w(H_0) \cap F_v(H_0)$ is a $(d-1)$-dimensional face or the two sets $F_w(H_0)$ and $F_v(H_0)$ are such that one is contained in the other.

Lemma 2.3. *Let K be an elementary fractal and let g be defined by $\boldsymbol{a} \in (0,1)^k$. Suppose $K_{iw} \sim K_{jv}$ for some $i \neq j \in \Sigma$, and $w, v \in \Sigma^*$ with $|w| = |v| \ge 1$. Then $\sigma(K_w) = K_v$ for some $\sigma \in G$, and $g(iw) = \frac{a_i}{a_j} g(jv)$.*

Proof. It is known that on a nested fractal, each element in P_0 belongs to exactly one n-cell for each n (Lindstrøm [29, IV.13 Proposition]). As a result, each n-cell contains at most one element of P_0 for each $n \ge 1$. By applying this property to K_i (or K_j), we see that $K_{iw} \cap K_{jv}$ is a singleton, denoted by $\{p\}$. Then there exist $p_1, p_2 \in P_0$ such that $F_i(p_1) = F_j(p_2) = p$. Let $\sigma \in G$ be the orthogonal reflection with respect to H_{p_1,p_2}. Then $\sigma(K_w) = K_v$ ([20, p.115]).

For the GSC, $K_{iw} \cap K_{jv}$ is a $(d-1)$-dimension face. Hence K_w and K_v are in the opposite face of H_0, and $\sigma(K_w) = K_v$ for a reflection on H_0.

The second part follows from $g(iw) = g(i)g(w)$ and $g(jv) = g(j)g(v) = g(j)g(w)$. □

For $\boldsymbol{a} \in (0,1)^k$, denote $a_* = \min\{a_i : 1 \le i \le k\}$, $a^* = \max\{a_i : 1 \le i \le k\}$. By Lemma 2.3, we obtain the following simple property which will be used frequently.

Proposition 2.4. *Let K be an elementary fractal, let $g := g_{\boldsymbol{a}}$, $\boldsymbol{a} \in (0,1)^k$ and let $c = a^*/a_*$. Suppose $K_w \sim K_v$ with $|w| = |v|$. Then we have*

$$c^{-1} g(v) \le g(w) \le c g(v).$$

Furthermore, if we reflect a chain γ contained in K_v along the appropriate hyperplane of $K_w \sim K_v$, and denote it by $R(\gamma)$, then $R(\gamma)$ is a chain contained in K_w and $c^{-1} g(\gamma) \le g(R(\gamma)) \le c g(\gamma)$.

Let Γ denote the class of chains $\gamma = (w(1), \dots, w(m))$ satisfying $K_{w(i)} \sim K_{w(i+1)}$ for all i. Similar to D_g in (1.2), define

$$D'_g(x,y) = \inf\{g(\gamma) : \gamma \in \Gamma \text{ connects } x \text{ and } y\}.$$

Clearly for the nested fractals, D'_g is just the same as the D_g.

Corollary 2.5. *For the GSC, we have $D'_g(\cdot,\cdot) \asymp D_g(\cdot,\cdot)$.*

Proof. By the non-diagonality assumption in the definition of GSC, we see that if $K_w \cap K_v \neq \emptyset$ with $|w| = n$, then there exists a chain $\gamma = \{w_1, \dots, w_m\}$ of n-cells such that $K_{w_i} \sim K_{w_{i+1}}$, $K_{w_1} = K_w$, $K_{w_m} \sim K_v$, and $m \le 2^d - 1$. By Proposition 2.4,

$$c^{-(2^d-2)} g(w_i) \le c^{-i+1} g(w_i) \le g(w), \quad 1 \le i \le m.$$

Hence we can replace the defining chains in $D_g(x,y)$ by the chains in $D'_g(x,y)$ and keep the above inequality. This yields $2^{-d}c^{-(2^d-2)}D'_g(x,y) \le D_g(x,y) \le D'_g(x,y)$ for all $x, y \in K$. □

Remark. For the GSC, the chains in Γ with $K_w \sim K_v$ are more convenient to use than $K_w \cap K_v \neq \emptyset$. We will use it without explicitly mentioning.

Denote by $\mathcal{M} = \{\boldsymbol{a} \in (0,1)^k : D_{g_{\boldsymbol{a}}}$ is a metric on $K\}$, and let $\mathcal{M}^c = (0,1)^k \setminus \mathcal{M}$. We call $\mathcal{M}$ the set of *admissible weights*, and $D_{g_{\boldsymbol{a}}}$ the *admissible metric* (determined by $\boldsymbol{a}$). If no confusion, we also say that g is a symmetric self-similar weight function to mean $g = g_{\boldsymbol{a}}$ for some $\boldsymbol{a} \in (0,1)^k$.

Lemma 2.6. *Suppose $\boldsymbol{a}, \boldsymbol{b} \in (0,1)^k$ and $\boldsymbol{b} \ge \boldsymbol{a}$ (coordinatewise). Then (i) $\boldsymbol{a} \in \mathcal{M}$ implies $\boldsymbol{b} \in \mathcal{M}$; (ii) $\boldsymbol{b} \in \mathcal{M}^c$ implies $\boldsymbol{a} \in \mathcal{M}^c$.*

Proof. It suffices to show that (ii) holds. Suppose that $\boldsymbol{b} \in \mathcal{M}^c$. By definition, there exist two distinct points $x, y \in K$ and a sequence of chains $\{\gamma_n\}_n$ between x and y such that $g_{\boldsymbol{b}}(\gamma_n) \to 0$ as $n \to \infty$. By assumption we have $g_{\boldsymbol{a}}(\gamma_n) \le g_{\boldsymbol{b}}(\gamma_n)$, hence $g_{\boldsymbol{a}}(\gamma_n) \to 0$ as $n \to \infty$, which implies that $\boldsymbol{a} \in \mathcal{M}^c$. □

The following is a crude estimation of $\mathcal{M}$ and $\mathcal{M}^c$.

Proposition 2.7. *(i) For a nested fractal K, there exist $0 < c \le C < 1$ such that $[C,1)^k \subset \mathcal{M}$ and $(0,c)^k \subset \mathcal{M}^c$; (ii) For a GSC, we have $[1/\ell,1)^k \subset \mathcal{M}$ and $(0,1/\ell)^k \subset \mathcal{M}^c$.*

Proof. (i) For two distinct $p, q \in P_0$, let $n(p,q)$ be the minimal length of the chains consisting of 1-cells between p and q. Let n_*, n^* be the minimum and maximum of $n(p,q)$ among all the pairs $p, q \in P_0$, respectively. As each $p \in P_0$ is contained in exactly one 1-cell (Lindstrøm [29]), therefore, we have $n^* \ge n_* \ge 2$.

Let g be the weight function generated by $\boldsymbol{a} = (1/n_*, \dots, 1/n_*)$. We show that $\boldsymbol{a} \in \mathcal{M}$, then by Lemma 2.6, $[1/n_*, 1)^k \subset \mathcal{M}$. The first part of (i) follows by letting $C = 1/n_*$.

For this, we let p and q be two distinct points in P_0. For any given simple chain γ between p and q (i.e., $K_{w(i)} \cap K_{w(j)} \neq \emptyset$ if and only if $|i-j| \le 1$), choose a cell K_w in γ such that w has the largest word length in γ. Let w' be the parent of w, and $J(w')$ be the set of the N children of w'. Then by using the definition of n_* on K_w, there exist at least n_* cells in $J(w')$ (including w) contained in γ. Observe that $g(u) = n_*^{-1} g(w')$ for any $u \in J(w')$, we have

$$g(w') \le \sum_{u \in J(w') \cap \gamma} g(u).$$

Hence if we replace the sub-chain of γ in $J(w')$ by w', then we get a new chain γ_* with $g(\gamma) \ge g(\gamma_*)$. We can also assume that γ_* is a simple chain by removing some cells in γ_*. By repeating this "merging" procedure to each cell with largest word length in the new chains, we finally get the trivial chain $\{\emptyset\}$. Thus $g(\gamma) \ge g(\emptyset) = 1$ and $D_g(p,q) \ge 1$. For arbitrary distinct two points $p, q \in K$, we can use a similar argument to show that $D_g(p,q) > 0$. Hence $(1/n_*, \dots, 1/n_*) \in \mathcal{M}$.

Next we show that $(0, 1/n^*)^k \subset \mathcal{M}^c$. Let $\boldsymbol{b} \in (0, 1/n^*)^k$ and fix any two distinct points p and q in P_0. For any $m \geq 0$, choose a chain γ_m of m-cells between p and q, where the length of γ_m is not larger than n^{*m}. Then $D_g(p,q) \leq (b^* n^*)^m$ ($b^* = \max_i\{b_i\}$). By letting $m \to \infty$, we have $D_g(p,q) = 0$ since $b^* < 1/n^*$. Hence we have $(0, 1/n^*)^k \subset \mathcal{M}^c$ and the second part of (i) follows by letting $c = 1/n^*$.

(ii) Let g be the weight function generated by $\boldsymbol{a} = (1/\ell, \dots, 1/\ell)$. Let p, q be two vertices of H_0 such that the line segment $\overline{pq}$ is parallel to an axis. Consider any chain γ between p and q. Since any cell with word length $m \geq 0$ has weight ℓ^{-m}, the projection of γ on the side $\overline{pq}$ covers $\overline{pq}$ so that $g(\gamma) \geq 1$.

Now let $x, y \in V_* = \bigcup_{w\in\Sigma^*} F_w(V_0)$ where V_0 is the set of vertices of H_0. Then x, y can be connected by finitely many line segments parallel to the axes. By self-similarity and the above, we have $D_{g_{\boldsymbol{a}}}(x,y) \geq |x-y|$. The density of V_* in K implies that $D_{g_{\boldsymbol{a}}}(x,y) > 0$ for all distinct $x, y \in K$. By Lemma 2.6, $[1/\ell, 1)^k \subset \mathcal{M}$.

To prove the last part, let $\boldsymbol{b} = (b_1, \dots, b_k)$ such that $b^* < 1/\ell$. Let p and q be two end points on a side of the cube H_0. Consider a simple chain γ_m with word length m connecting p and q and along the edge of the GSC. Then we have $g(\gamma_m) \leq \ell^m (b^*)^m \to 0$ as $m \to \infty$. Therefore $D_g(p,q) = 0$ and $\boldsymbol{b} \notin \mathcal{M}$. It follows that $(0, 1/\ell)^k \subset \mathcal{M}^c$. □

Let $g := g_{\boldsymbol{a}}$ be the weight function on K with $\boldsymbol{a} = (a_1, \dots, a_k) \in (0,1)^k$. Define a curve $\boldsymbol{a}(\lambda) : (0,\infty) \to (0,1)^k$ by

$$\boldsymbol{a}(\lambda) = (a_1^\lambda, \dots, a_k^\lambda), \quad \lambda \in (0,\infty).$$

Clearly $\boldsymbol{a}(1) = \boldsymbol{a}$. By Proposition 2.7, $\boldsymbol{a}(\lambda) \in \mathcal{M}$ for λ small enough, and $\boldsymbol{a}(\lambda) \in \mathcal{M}^c$ for λ large enough. Since $\boldsymbol{a}(\lambda_1) > \boldsymbol{a}(\lambda_2)$ (coordinatewise) for $\lambda_1 < \lambda_2$, by Lemma 2.6, $\boldsymbol{a}(\lambda_2) \in \mathcal{M}$ implies $\boldsymbol{a}(\lambda_1) \in \mathcal{M}$. This yields a unique *boundary point* $\lambda_{\boldsymbol{a}} > 0$ such that $\boldsymbol{a}(\lambda) \in \mathcal{M}$ if $\lambda > \lambda_{\boldsymbol{a}}$, and $\boldsymbol{a}(\lambda) \in \mathcal{M}^c$ if $\lambda < \lambda_{\boldsymbol{a}}$ (see Figure 1). Denote by $\Lambda_k = \{\boldsymbol{a} \in (0,1)^k : \sum_{i=1}^k a_i = 1\}$ the set of normalized vectors of $(0,1)^k$. Let

$$S = \{\boldsymbol{a}(\lambda_{\boldsymbol{a}}) : \boldsymbol{a} \in \Lambda_k\}.$$

It follows that S separates $(0,1)^k$ into two connected components $\mathcal{M}$ and $\mathcal{M}^c$. We call S the *boundary surface* of $\mathcal{M}$.

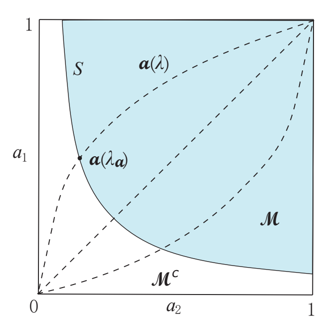

FIGURE 1. $\mathcal{M}$, $\mathcal{M}^c$ and S

Corollary 2.8. *With the above notations, let $\partial\mathcal{M}$ denote the boundary of $\mathcal{M}$ in $(0,1)^k$. Then $S = \partial\mathcal{M}$.*

Proof. It is clear that $S \subseteq \partial\mathcal{M}$. To prove the reverse inclusion, let $\boldsymbol{b} \in \partial\mathcal{M}$. Suppose $\boldsymbol{b} \notin S$. Then there is an $\boldsymbol{a} \in \Lambda_k$ such that $\boldsymbol{b} = \boldsymbol{a}(\lambda)$, and $\boldsymbol{a}(\lambda_{\boldsymbol{a}}) \in S$ and $\lambda \neq \lambda_{\boldsymbol{a}}$. If $\lambda < \lambda_{\boldsymbol{a}}$, let λ_1 be such that $\lambda < \lambda_1 < \lambda_{\boldsymbol{a}}$, then $\boldsymbol{a}(\lambda_1) \in \mathcal{M}$. By Lemma 2.6(i), we see that $\mathcal{M}$ contains all $\boldsymbol{b}' > \boldsymbol{a}(\lambda_1)$, which is a subset of $\mathcal{M}^o$, the interior of $\mathcal{M}$. Hence $\boldsymbol{b} = \boldsymbol{a}(\lambda) \in \mathcal{M}^o$. This contradicts that $\boldsymbol{b} \in \partial\mathcal{M}$. If $\lambda_{\boldsymbol{a}} < \lambda$, then we can use a similar method (apply Lemma 2.6(ii)) to obtain a similar contradiction. □

3. A dichotomy for $D_g(\cdot,\cdot)$

Note that in general, for any weight function g, either D_g is a metric or there exists a pair $x \neq y$ in K such that $D_g(x, y) = 0$. In this section we prove a stronger conclusion for symmetric self-similar weights on the elementary fractals.

Theorem 3.1. *Let K be an elementary fractal, and let g be a symmetric self-similar weight function. Then D_g is either a metric or identically 0 on K.*

Equivalently, the theorem says that $D_{g_{\boldsymbol{a}}} \equiv 0$ for $\boldsymbol{a} \in \mathcal{M}^c$. Since the proof involves different symmetries in the nested fractals and the GSC, we divide the proofs into two separate parts.

Lemma 3.2. *Let K be a nested fractal, and let P_0 be the set of essential fixed points. Suppose $D_g(q^*, s^*) = 0$ for some distinct $q^*, s^* \in P_0$, then $D_g(q, s) = 0$ for all $q, s \in P_0$.*

Proof. We define an equivalence relation in P_0 as follows: for any two points q and s in P_0, we write $q \sim s$ if either $q = s$ or there is a finite sequence of points $\{q_i\}_{i=0}^{l}$ in P_0 with $q_0 = q$ and $q_l = s$, and for each $0 \leq i \leq l-1$, there is a $\sigma_i \in G$ satisfying $\sigma_i(q_i) = q^*$ and $\sigma_i(q_{i+1}) = s^*$. It is easy to check that "$\sim$" is indeed an equivalence relation on P_0, which is invariant under G, that is, for any points $q, s \in P_0$, $\sigma \in G$, if $q \sim s$, then $\sigma(q) \sim \sigma(s)$. Obviously, by using the triangle inequality of D_g, we have $D_g(q, s) = 0$ for any $q \sim s$.

If $q, s \in P_0$ are two distinct points, let $R_{q,s}$ be the orthogonal reflection along $H_{q,s} = \{z \in \mathbb{R}^d : |q - z| = |s - z|\}$. Let H_q, H_s be the closed half-space containing q, s respectively. Then by Sabot [31, Lemma 6.4], for any G-invariant equivalent relation on P_0, any equivalent class intersects both half-spaces. Hence there is a point q' in $P_0 \cap H_s$ with $q \sim q'$, where "$\sim$" is the relation defined as above. Therefore we have $D_g(q, q') = 0$, and there is a sequence of chains $\{\eta_m\}_{m\geq 0}$ between q and q' with the total weight $g(\eta_m)$ tending to 0 as $m \to \infty$.

For $m \geq 0$, let η'_m be the chain obtained by reflecting the part of the cells in η_m contained in H_q to H_s by $R_{q,s}$. Then η'_m connects q' and s. By the symmetry of the weight function g, we have $g(\eta'_m) = g(\eta_m) \to 0$ as $m \to \infty$, which yields $D_g(q', s) = 0$. Then $D_g(q, s) \leq D_g(q, q') + D_g(q', s) = 0$ and the lemma holds. □

Proof of Theorem 3.1 for nested fractals. Let x_0, y_0 be two distinct points in K with $D_g(x_0, y_0) = 0$. Without loss of generality, assume K is the smallest subcell which contains both x_0 and y_0. Then there exist distinct $i, j \in \{1, 2, \dots, N\}$ such that $x_0 \in K_i$ and $y_0 \in K_j$. Let $E_{y_0} = \bigcup_{u \in \Sigma:\, y_0 \in K_u} F_u(P_0)$. Then E_{y_0} is a finite set and $x_0 \notin E_{y_0}$.

Let $K_v, v \in \Sigma^*$, be a cell containing x_0 with word length large enough such that $K_v \cap E_{y_0} = \emptyset$. Let $n = |v|$ and $m_0 \in \mathbb{Z}^+$ satisfying $m_0^{-1} < a_*^n$. From $D_g(x_0, y_0) = 0$, for each positive integer $m \geq m_0$, there exists a chain γ_m between x_0, y_0 such that $g(\gamma_m) < m^{-1}$. As $m^{-1} < a_*^n$, every cell in γ_m has length $> n$. By the nesting property, the chain γ_m must pass through one of the points in $F_v(P_0)$, and one of the points in E_{y_0}. By the finiteness of both $\#F_v(P_0)$ and $\#E_{y_0}$, there is a subsequence of $\{\gamma_m\}$, still denoted by $\{\gamma_m\}$, and $q^* \in F_v(P_0)$, $\tilde{s} \in E_{y_0}$, such that each γ_m passes through both q^* and $\tilde{s}$. Let $\{\widetilde{\gamma}_m\}$ be sub-chains of $\{\gamma_m\}$ connecting q^* and $\tilde{s}$. Let

$$E_{q^*} = \bigcup_{w \in \Sigma^n:\, q^* \in K_w} F_w(P_0) \setminus \{q^*\}.$$

By taking subsequence and sub-chains again, we can find a point $s^* \in E_{q^*}$, a word $w \in \Sigma^n$, and a sub-chain γ'_m of $\widetilde{\gamma}_m$ between q^* and s^*, contained in K_w. Since our choices of sub-chains $\widetilde{\gamma}_m$ and γ'_m of γ_m do not increase the total weight, we have $D_g(q^*, s^*) = 0$.

Using self-similarity, we can dilate q^*, s^* to be two distinct points in P_0. By Lemma 3.2, we have $D_g(q, s) = 0$ for all $q, s \in P_0$. Hence by self-similarity, $D_g(q, s) = 0$ for all $q, s \in F_u(P_0), u \in \Sigma^*$. In general, for any two points q, s in K, we can use the approximation by points in $V_* = \bigcup_{u \in \Sigma^*} F_u(P_0)$ to show that $D_g(q, s) = 0$. This completes the proof. □

We then turn to the GSC. For any point $p \in \mathbb{R}^d$, we denote by $x_i(p)$ the i-th coordinate of p with $1 \leq i \leq d$. For any subset E of $\mathbb{R}^d$, we denote by $\pi_i(E)$ the orthogonal projection of E onto the i-th axis, i.e. $\pi_i(E) = \{x_i(p) : p \in E\}$. We will use $K_{w(i)} \sim K_{w(i+1)}$ for the connection of a chain (see Corollary 2.5). Similar to Lemma 3.2, we have the following for the GSC.

Lemma 3.3. *Let K be a GSC. Suppose q^* and s^* are on the two opposite faces of the cube $H_0 = [0, 1]^d$ and $D_g(q^*, s^*) = 0$, then $D_g(q, s) = 0$ for all q, s in the vertices of H_0 with $\overline{qs}$ parallel to one of the coordinate axes.*

Proof. We assume that $x_1(q^*) = 0$, $x_1(s^*) = 1$. Define q', s' by changing the first coordinates of q^*, s^*: $x_1(q') = 1$, $x_1(s') = 0$ and $x_i(q') = x_i(q^*)$, $x_i(s') = x_i(s^*)$ for $2 \leq i \leq d$. By symmetry, $D_g(q', s') = 0$.

For $0 \leq j \leq \ell$, let p_j be the point in K with coordinates $x_1(p_j) = j/\ell$, and $x_i(p_j) = 0$ for $2 \leq i \leq d$. For $1 \leq j \leq \ell$, let $F_j : H_0 \to H_0$ be such that $F_j(x) = x/\ell + p_{j-1}$. From the borders included condition of GSC in Definition 2.2, $\{F_j\}_{1 \leq j \leq \ell}$ is a subset of $\{F_Q\}_{Q \in \mathcal{S}}$ such that each cube $F_j(H_0)$ locates along the line segment $\overline{p_0 p_\ell}$. It follows that $D_g(F_j(q^*), F_j(s^*)) = D_g(F_j(q'), F_j(s')) = 0$ for $1 \leq j \leq \ell$.

Notice that $F_j(s^*) = F_{j+1}(s')$ and $F_j(q') = F_{j+1}(q^*)$ for $1 \le j < \ell$. If ℓ is an odd number, we have

$$\begin{aligned} &D_g(F_1(q^*), F_\ell(s^*)) \\ \le\ &D_g(F_1(q^*), F_1(s^*)) + D_g(F_1(s^*), F_2(q')) + \cdots + D_g(F_{\ell-1}(q'), F_\ell(s^*)) \\ =\ &D_g(F_1(q^*), F_1(s^*)) + D_g(F_2(s'), F_2(q')) + \cdots + D_g(F_\ell(q^*), F_\ell(s^*)) = 0. \end{aligned}$$

By using this repeatedly, we see that $D_g(F_{1^n}(q^*), F_{\ell^n}(s^*)) = 0$ for all integers $n \ge 0$. By the continuity of D_g, we have $D_g(p_0, p_\ell) = 0$. Similarly, if ℓ is even, by considering $D_g(F_{1^n}(q^*), F_{\ell^n}(q^*))$ instead, we also have $D_g(p_0, p_\ell) = 0$. The lemma follows by symmetry. □

Proof of Theorem 3.1 for the GSC. Let q_0, s_0 be two distinct points in K with $D_g(q_0, s_0) = 0$. We select an $n > 0$ such that $2 \cdot \ell^{-n} < \max\{|x_i(q_0) - x_i(s_0)| : 1 \le i \le d\}$. Without loss of generality, we assume that $x_1(s_0) > x_1(q_0) + 2 \cdot \ell^{-n}$. Define $\alpha_n = \lceil \ell^n x_1(q_0) \rceil \cdot \ell^{-n}$, where $\lceil t \rceil$ is the minimal integer no smaller than t.

Let $m_0 \in \mathbb{Z}^+$ satisfy $m_0^{-1} < a_*^n$. From $D_g(q_0, s_0) = 0$, then for $m \ge m_0$, there exists a chain γ_m between q_0 and s_0 such that $g(\gamma_m) < m^{-1}$. From $m^{-1} < a_*^n$, every cell in γ_m has length greater than n. Therefore, we can pick two points q_m and s_m in K, and a sub-chain $\widetilde{\gamma}_m$ of γ_m between q_m and s_m, such that $x_1(q_m) = \alpha_n$, $x_1(s_m) = \alpha_n + \ell^{-n}$, and $\pi_1(\cup\widetilde{\gamma}_m) = [\alpha_n, \alpha_n + \ell^{-n}]$. By taking subsequence, we can find two n-cells K_w and K_v (independent of m) such that $q_m \in K_w$ and $s_m \in K_v$, and $\pi_1(K_w) = \pi_1(K_v) = [\alpha_n, \alpha_n + \ell^{-n}]$.

For each $m \ge m_0$, we replace each cell K_u in $\widetilde{\gamma}_m$ by $K_{u|_n}$ and delete the repeated ones to obtain a chain η_m consisting of n-cells. Then obviously, the chain η_m starts from K_w and ends with K_v, and $\pi_1(\cup\eta_m) = [\alpha_n, \alpha_n + \ell^{-n}]$. Also for every two successive cells in η_m, they share a same $(d-1)$-dimensional hyperplane which is always parallel to the 1-st coordinate axis. Reflecting the chain $\widetilde{\gamma}_m$ according to these $(d-1)$-dimensional hyperplanes along η_m from K_w to K_v successively, we obtain a new chain γ'_m contained in K_v (see Figure 2). Note that there is a point q'_m in K_v with $x_1(q'_m) = \alpha_n$ such that γ'_m is between q'_m and s_m. By Proposition 2.4,

$$g(\gamma'_m) \le c^{\ell^{(d-1)n}} g(\widetilde{\gamma}_m) \le c^{\ell^{(d-1)n}} g(\gamma_m) < c^{\ell^{(d-1)n}} m^{-1}, \qquad \text{where } c = a^*/a_*.$$

By taking subsequence if necessary, we may assume that q'_m converges to q^*, and s_m converges to s^*. Then $q^*, s^* \in K_v$ with $x_1(q^*) = \alpha_n$, $x_1(s^*) = \alpha_n + \ell^{-n}$, and $D_g(q^*, s^*) = 0$.

Using self-similarity, we can dilate q^*, s^* to the two opposite faces of H_0. Then by Lemma 3.3, $D_g(q, s) = 0$ for all q, s in the vertices of H_0 with $\overline{qs}$ parallel to one of the coordinate axes. Since any two points in V_* can be connected by finitely many pairs of vertices of $F_w(H_0)$, by using the self-similarity and the triangle inequality of D_g, we must have that $D_g(q, s) = 0$ for any $q, s \in V_*$. By using the continuity of D_g w.r.t. the Euclidean metric, we have that $D_g(q, s) = 0$ for all $q, s \in K$. This completes the proof. □

We will call a procedure on a chain γ *splitting* if it splits the cells in γ to obtain a finer chain γ' such that $g(\gamma') \le g(\gamma)$.

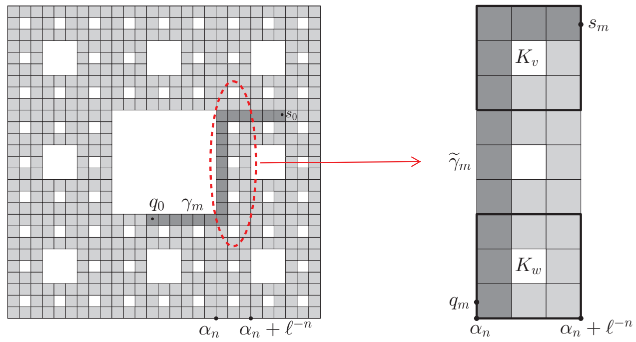

FIGURE 2. Chains γ_m and $\widetilde{\gamma}_m$

Lemma 3.4. *Let K be an elementary fractal. Assume that D_g is not a metric on K, then there exists $N_0 > 0$, such that for any two points x and y in K, there is a sequence of chains $\{\gamma_n\}_{n=0}^{\infty}$ between x and y, such that each chain is a splitting of the previous one, and*

(i) $g(\gamma_n) \leq 1/4^{n+1}$;
(ii) $nN_0 < |u| \leq (n+1)N_0$ *for any $u \in \gamma_n$.*

Proof. We first claim that there exists a positive integer N_0, such that for any two points x and y in K, there is a chain $\gamma_{x,y}$ between x and y such that the following two conditions hold:

$$g(\gamma_{x,y}) \leq 1/4, \quad \text{and} \quad 0 < |u| \leq N_0, \;\; \forall\, u \in \gamma_{x,y}. \tag{3.1}$$

Indeed, let N_1 be the smallest integer such that $a^{*N_1} \leq 1/16$. Let K_w and K_v be two cells with $|w| = |v| = N_1$, $x \in K_w$ and $y \in K_v$. In the case that $K_w \cap K_v \neq \emptyset$, we define $\gamma_{w,v} = \{w, v\}$ so that

$$g(\gamma_{w,v}) = g(w) + g(v) \leq 2a^{*N_1} \leq 1/8.$$

In the case that $K_w \cap K_v = \emptyset$, by Theorem 3.1, there is a chain $\eta_{w,v}$ connecting K_w and K_v such that $g(\eta_{w,v}) \leq 1/8$. Let $\gamma_{w,v}$ be the chain constructed by adding $\eta_{w,v}$ in between w and v. Then we have

$$g(\gamma_{w,v}) = g(w) + g(\eta_{w,v}) + g(v) \leq 1/4.$$

Now set $N_{w,v} = \max\{|u| : u \in \gamma_{w,v}\}$, and let N_0 be the maximum of $N_{w,v}$ among all the pairs w, v in Σ^{N_1}. Then for all $\gamma_{w,v}$, $g(\gamma_{w,v}) \leq 1/4$ and $\gamma_{w,v}$ consists of u such that $0 < |u| \leq N_0$, and the claim follows. For simplicity, we write $\gamma_0 := \gamma_{x,y} := (w(1), \ldots, w(m))$.

We now construct γ_1. For each word $w(i)$ in γ_0, we perform a splitting as follows. Let $x' \in K_{w(i)} \cap K_{w(i-1)}$ and $y' \in K_{w(i)} \cap K_{w(i+1)}$ (if $i = 1$, we just take $w(0) = w(1)$ and $x = x'$, and similar for $i = m$). For each $w(i)$, consider the pullback $F_{w(i)}^{-1}(K_{w(i)})(= K)$, we apply the claim to $x'' = F_{w(i)}^{-1}(x')$ and $y'' = F_{w(i)}^{-1}(y')$ to obtain a chain $\gamma_{x'',y''}$ satisfying (3.1). Consider $F_{w(i)}(\gamma_{x'',y''})$, which is a chain between x' and y' in $K_{w(i)}$ consists of cells $F_{w(i)}(K_u)$ for each $u \in \gamma_{x'',y''}$. By self-similarity of g and (3.1), the chain has the following property:

$$g(F_{w(i)}(\gamma_{x'',y''})) = \sum_{u \in F_{w(i)}(\gamma_{x'',y''})} g(u) = \sum_{u \in \gamma_{x'',y''}} g(u) \cdot g(w(i)) \le (1/4) \cdot g(w(i)),$$

with $|w(i)| < |u| \le N_0 + |w(i)|$ for all $u \in F_{w(i)}(\gamma_{x'',y''})$.

Now we replace the word $w(i)$ in γ_0 by the chain $F_{w(i)}(\gamma_{x'',y''})$ for each i, and obtain a new chain. We keep doing the same splitting for words with length $\le N_0$ in the new chain. After finite many times, we obtain a chain γ_1 between x and y such that each word in γ_1 has length $> N_0$. Since we are using the claim to do the splitting, each word in γ_1 has length $\le 2N_0$ (Indeed, at the beginning of splitting, $|w(i)| \le N_0$, and after the splitting, the new words has length $\le N_0 + N_0 = 2N_0$). With all these,

$$g(\gamma_1) \le (1/4) \cdot g(\gamma_0) \le 1/4^2, \text{ with } N_0 < |u| \le 2N_0, \ \forall\, u \in \gamma_1.$$

Inductively, we adopt the same procedure to construct γ_{n+1} from γ_n: for each word $w_i \in \gamma_n$, we use the same pull-back technique to bring $K_{w(i)}$ to K, and apply the claim to carry out the splitting, and the lemma follows. □

Proposition 3.5. *For an elementary fractal K, $\mathcal{M}^c$ is an open set in* $(0,1)^k$.

Proof. We adopt the same notation as in Lemma 3.4. Let $\boldsymbol{a} \in \mathcal{M}^c$ and $\varepsilon > 0$. Consider the weight function $g^{(\varepsilon)}$ which is defined by the vector $\boldsymbol{a}(\varepsilon) = (a_1 + \varepsilon_1, a_2 + \varepsilon_2, \dots, a_k + \varepsilon_k)$, with $|\varepsilon_i| \le \varepsilon$ for $i = 1, 2, \dots, k$. Let $\alpha_\varepsilon = 1 + \frac{\varepsilon}{a_*}$, and let γ_n be as in the lemma, then $|w| \le (n+1)N_0$ for all $w \in \gamma_n$. It follows that

$$\begin{aligned} g^{(\varepsilon)}(\gamma_n) &= \sum_{w \in \gamma_n} g^{(\varepsilon)}(w) = \sum_{w \in \gamma_n} g(w) \frac{g^{(\varepsilon)}(w)}{g(w)} \le \sum_{w \in \gamma_n} g(w)\, \alpha_\varepsilon^{|w|} \\ &\le g(\gamma_n)\, \alpha_\varepsilon^{(n+1)N_0} \le \left(4^{-1} \alpha_\varepsilon^{N_0}\right)^{n+1}, \quad \forall n \ge 0. \end{aligned}$$

Choose $\varepsilon > 0$ small enough such that $4^{-1}\alpha_\varepsilon^{N_0} \le \frac{1}{2}$, then $g^{(\varepsilon)}(\gamma_n) \to 0$ as $n \to \infty$. This implies that $\boldsymbol{a}(\varepsilon) \in \mathcal{M}^c$ and thus $\mathcal{M}^c$ is open. □

4. Metric chain condition (MCC)

It follows from Proposition 3.5 that $\mathcal{M}$ is closed in $(0,1)^k$. As S is the boundary surface of $\mathcal{M}$ (Corollary 2.8), we have $S \subset \mathcal{M}$ and $\mathcal{M} \setminus S = \mathcal{M}^o$, the interior of $\mathcal{M}$. In this section, we will study $D_{g_{\boldsymbol{a}}}$ for $\boldsymbol{a} \in \mathcal{M}$, and in particular in S in connection with the MCC (Definition 1.5).

According to Kigami [22], we say that D_g is 1-*adapted* to g if there exists a constant $C > 0$ such that for all $x, y \in K$,

$$(\ D_g(x,y) \le)\ \inf\left\{g(\gamma) : \gamma \text{ connects } x \text{ and } y, |\gamma| \le 2\right\} \le C D_g(x,y). \tag{4.1}$$

Remark 1. In [22], the terminology of "m-adapted" is defined for any integer $m \ge 1$ by replacing the "$|\gamma| \le 2$" in (4.1) by "$|\gamma| \le 1 + m$". For $g = g_{\boldsymbol{a}}$, we can actually use chains of the form $\gamma = \{w, v\}$ with $|w| = |v|$ to connect x, y. Indeed let $x \in K_w, y \in K_v$, and assume that they do not contain each other. Suppose $|w| > |v|$ (or $|w| < |v|$), we truncate the last indices of w to w' so that $|w'| = |v|$. By Proposition 2.4 and the proof in Corollary 2.5, we have $c^{-1}g(v) \le g(w') \le cg(v)$ where c depends on $\boldsymbol{a}$ and d. Hence (4.1) still holds with the constant $C' = (1+c)C$.

As a special case of [22, Theorem 2.3.16], we have

Proposition 4.1. *Let $\boldsymbol{a} \in \mathcal{M}$ and g be the associated weight function. Then the metric D_g is 1-adapted to g.*

To study the MCC on K, we use the n-chains defined in the following. For $n \ge 0$, a chain γ is called an *n-chain* if all the words in γ have length equal to n. Define

$$D_g^{(n)}(x, y) := \min\left\{g(\gamma) | \, \gamma \text{ is an } n\text{-chain between } x \text{ and } y\right\}.$$

Clearly, $D_g(x, y) \le D_g^{(n)}(x, y)$ for all $x, y \in K$.

Remark 2. We will need the following simple fact: for $\boldsymbol{a} \in \mathcal{M}^c$,

$$\lim_{n\to\infty} D_{g_{\boldsymbol{a}}}^{(n)}(x, y) = D_{g_{\boldsymbol{a}}}(x, y) = 0, \quad \forall\, x, y \in K. \tag{4.2}$$

Indeed, for a given n, let m be the smallest integer such that $n \le mN_0$. By Lemma 3.4, there exists a chain γ_m connecting x, y such that $g(\gamma_m) \le 4^{-m-1}$, and $mN_0 < |u| \le (m + 1)N_0$ for $u \in \gamma_m$. Now we define an n-chain $\gamma^{(n)}$ by truncating each $u \in \gamma_m$ to u' of size n. Hence the length of each word u' in $\gamma^{(n)}$ has length differ from u by at most $2N_0$. This implies $g_{\boldsymbol{a}}(\gamma^{(n)}) \le C4^{-m}$ for a constant C so that (4.2) holds.

First we prove

Proposition 4.2. *Let K be an elementary fractal, and let $\boldsymbol{a} \in \mathcal{M}^o$ (the interior of $\mathcal{M}$). Then*

$$\lim_{n\to\infty} D_{g_{\boldsymbol{a}}}^{(n)}(x, y) = \infty, \quad \forall\, x \ne y \in K.$$

In this case, $D_{g_{\boldsymbol{a}}}$ is a metric but does not satisfy the MCC.

Proof. For $\boldsymbol{a} \in \mathcal{M}^o$, first we claim that for any distinct $x, y \in K$, there exist $C > 0$ and $\sigma > 1$ such that for all $n \ge 0$,

$$D_{g_{\boldsymbol{a}}}^{(n)}(x, y) \ge C\sigma^n.$$

Indeed, since $\boldsymbol{a} \in \mathcal{M}^o$, there exist $\boldsymbol{b} \in \Lambda_k$ and $0 < \lambda < \lambda_{\boldsymbol{b}}$, such that $\boldsymbol{a} = \boldsymbol{b}(\lambda)$. Denote by $\delta = \lambda_{\boldsymbol{b}} - \lambda$ and let g_0 be the weight function of $\boldsymbol{b}(\lambda_{\boldsymbol{b}})$. Since $\boldsymbol{b}(\lambda_{\boldsymbol{b}}) \in S \subset \mathcal{M}$, we have $D_{g_0}(x, y) > 0$ and for each chain γ between x and y,

$$g_{\boldsymbol{a}}(\gamma) = \sum_{w\in\gamma} \left(g_{\boldsymbol{b}}(w)\right)^{\lambda} = \sum_{w\in\gamma} \left(g_{\boldsymbol{b}}(w)\right)^{\lambda_{\boldsymbol{b}}} \cdot \left(g_{\boldsymbol{b}}(w)\right)^{-\delta},$$

and hence

$$D_{g_{\boldsymbol{a}}}^{(n)}(x, y) \ge b^{*-\delta n} \cdot D_{g_0}^{(n)}(x, y) \ge (b^{*-\delta})^n \cdot D_{g_0}(x, y).$$

This proves the claim, and clearly implies that $\lim_{n\to\infty} D_{g_{\boldsymbol{a}}}^{(n)}(x, y) = \infty$.

To prove that $D_{g_{\boldsymbol{a}}}$ does not satisfy the MCC, we assume the contrary. We write D_g for $D_{g_{\boldsymbol{a}}}$ for simplicity. For two distinct points x and y, there is $C > 0$ such that for any integer $n \geq 1$, there is a sequence $x = x_0, x_1, \ldots, x_n = y$ such that

$$D_g(x_i, x_{i+1}) \leq Cn^{-1} D_g(x, y), \quad \text{for } 0 \leq i \leq n-1. \tag{4.3}$$

Pick $\lambda > 1$ close to 1 such that $D_{g^{(\lambda)}}$ is a metric, where $D_{g^{(\lambda)}}$ is given by the weight $\boldsymbol{a}(\lambda) = (a_1^\lambda, a_2^\lambda, \ldots, a_k^\lambda)$. By using the 1-adaptedness of D_g and $D_{g^{(\lambda)}}$ (Proposition 4.1), we have

$$D_{g^{(\lambda)}}(x, y) \asymp \left(D_g(x, y)\right)^\lambda, \quad D_{g^{(\lambda)}}(x_i, x_{i+1}) \asymp \left(D_g(x_i, x_{i+1})\right)^\lambda, \ \ \forall\, i.$$

By using this, triangle inequality and (4.3), it follows that

$$\begin{aligned} D_{g^{(\lambda)}}(x, y) &\leq \sum_{i=0}^{n-1} D_{g^{(\lambda)}}(x_i, x_{i+1}) \leq C' \sum_{i=0}^{n-1} \left(D_g(x_i, x_{i+1})\right)^\lambda \\ &\leq C'' \sum_{i=0}^{n-1} \left(n^{-1} D_g(x, y)\right)^\lambda \leq C'' n^{1-\lambda} \left(D_g(x, y)\right)^\lambda. \end{aligned}$$

Letting $n \to \infty$, we have $D_{g^{(\lambda)}}(x, y) = 0$, a contradiction. Hence D_g does not satisfy the MCC. □

The main purpose of the section is to prove Theorem 4.4. We need a lemma.

Lemma 4.3. *Let K be an elementary fractal, $x, y \in K$ and $\boldsymbol{a} \in S$. Then*

$$\sup_{n \geq 0} D_{g_{\boldsymbol{a}}}^{(n)}(x, y) < \infty.$$

Moreover, there are n_0 (depends on x, y) and $C > 0$ (depends on $\boldsymbol{a}$) such that

$$\sup_{n \geq n_0} D_{g_{\boldsymbol{a}}}^{(n)}(x, y) \leq C D_{g_{\boldsymbol{a}}}(x, y). \tag{4.4}$$

Proof. We first proof the case for the nested fractals. For $p, q \in P_0$, we denote by $p \bowtie q$ if $\sup_{n \geq 0} D_g^{(n)}(p, q) < \infty$. This gives an equivalent relation on P_0 which is preserved under the group G. Observe that by using a similar argument as in Lemma 3.2, we conclude that $p \bowtie q$ for some distinct $p, q \in P_0$ if and only if $p \bowtie q$ for all $p, q \in P_0$.

We first show that $p \bowtie q$ for all $p, q \in P_0$. Suppose otherwise, $\sup_{n \geq 0} D_g^{(n)}(p, q) = \infty$ for all distinct $p, q \in P_0$, choose $N_0 \geq 1$ such that $D_g^{(N_0)}(p, q) \geq 2$ for all $p \neq q \in P_0$. It follows that for each $m \geq 1$,

$$D_g^{(mN_0)}(p, q) \geq 2^m, \qquad \forall p \neq q \in P_0. \tag{4.5}$$

Indeed, let $\gamma = \{w(1), \ldots, w(t)\}$ be a chain connecting p, q where $|w(i)| = (m+1)N_0$ for all i. By considering $w(i)|_{N_0}$, we obtain a sequence of N_0-cells $\{K_{u(j)}\}_{j=1}^s$ connecting p, q, and decompose γ into sub-chains $\{\gamma_j\}_{j=1}^s$ with γ_j contained in $K_{u(j)}$. Then $F_{u(j)}^{-1}(\gamma_j)$ is an mN_0-chain, and induction implies $g(F_{u(i)}^{-1}(\gamma_i)) \geq 2^m$, so that $g(\gamma) \geq \sum g(u(i)) 2^m \geq 2^{m+1}$ and $D_g^{((m+1)N_0)}(p, q) \geq 2^{m+1}$ for all $p \neq q \in P_0$.

Choose $\varepsilon > 0$ small enough such that $(a_*)^{\varepsilon N_0} \geq 1/2$. Let $\boldsymbol{a}(\varepsilon) = (a_1^{1+\varepsilon}, \ldots, a_k^{1+\varepsilon})$, and $g^{(\varepsilon)}$ be the weight function given by $\boldsymbol{a}(\varepsilon)$. Then for any mN_0-chain $\gamma^{(mN_0)}$ between p and q, we have

$$g^{(\varepsilon)}(\gamma^{(mN_0)}) = \sum_{w\in\gamma^{(mN_0)}} g^{(\varepsilon)}(w) = \sum_{w\in\gamma^{(mN_0)}} g(w)^{1+\varepsilon} \geq \sum_{w\in\gamma^{(mN_0)}} g(w)(a_*)^{\varepsilon m N_0} \geq \frac{g(\gamma^{(mN_0)})}{2^m}.$$

Combining this with (4.5), we see that for all $m \geq 1$, $D^{(mN_0)}_{g^{(\varepsilon)}}(p,q) \geq 1$. On the other hand, it follows from $\boldsymbol{a} \in S$ that $\boldsymbol{a}(\varepsilon) \in \mathcal{M}^c$, which gives $\lim_{n\to\infty} D^{(n)}_{g^{(\varepsilon)}}(p,q) = 0$ by Remark 2, a contraction. This proves $p \bowtie q$ for all $p, q \in P_0$.

To complete the proof of the first part of the lemma, we show further that $\sup_{n\geq 0} D_g^{(n)}(x,y) < \infty$ for any $x, y \in K$. Indeed, we can find a sequence of cells $\{K_{w(i)}\}_{i\geq 0}$ such that $w(0) = \emptyset$ and $w(i+1)$ is a child of $w(i)$, and $\bigcap_{i\geq 1} K_{w(i)} = \{x\}$. Pick an arbitrary $x_i \in F_{w(i)}(P_0)$, we obtain a sequence of points $\{x_i\}_{i=0}^\infty$, such that $\lim_{i\to\infty} x_i = x$. We also see that x_i and x_{i+1} can be connected by a uniformly bounded number of pairs in $F_{w(i+1)}(P_0)$. Let $M = \max_{p,q\in P_0} \sup_{n\geq 0} D_g^{(n)}(p,q)$. Hence by self-similarity, there exists a constant $C > 0$, such that for all i,

$$\sup_{n\geq 0} D_g^{(n)}(x_i, x_{i+1}) \leq C g(w(i+1)) \cdot M.$$

By summing up over i, and observe that $\sum_{i=1}^\infty g(w(i)) \leq \sum_{m=1}^\infty a^{*m} < \infty$, we have

$$\sup_{n\geq 0} D_g^{(n)}(x_0, x) \leq C \sum_{i=1}^\infty g(w(i)) \cdot M \leq C' M.$$

Similarly, we pick $\{y_i\}_{i=0}^\infty$, such that $\lim_{i\to\infty} y_i = y$ with $y_0 = x_0$. Then we have $\sup_{n\geq 0} D_g^{(n)}(x_0, y) \leq C'M$, so that $\sup_{n\geq 0} D_g^{(n)}(x,y) \leq 2C'M < \infty$. This proves the first part.

To prove the last assertion, we observe that in the 1-adaptedness of D_g, Remark 1 allows us to assume that the two-word chain $\gamma = \{w, v\}$ connecting x, y is such that $|w| = |v|$. Let

$$n_0 = n_0(x,y) := \max\{n \geq 0 : \ \gamma = \{w,v\} \text{ connecting } x, y \text{ and } |w| = |v| = n\}.$$

Clearly $n_0 < \infty$. We can assume that $\gamma = \{w, v\}$ attains the maximum. Let $z \in K_w \cap K_v$. Since the two points x and z are both in K_w, consider $F_w^{-1}(x), F_w^{-1}(z) \in K$. By the first conclusion of the lemma, we have

$$\sup_{n\geq 0} D_g^{(n)}\left(F_w^{-1}(x), F_w^{-1}(z)\right) \leq C_0.$$

It follows from the self-similarity that $\sup_{n\geq n_0} D_g^{(n)}(x,z) \leq C_0 g(w)$. Similarly, $\sup_{n\geq n_0} D_g^{(n)}(y,z) \leq C_0 g(v)$. Hence we have

$$\sup_{n\geq n_0} D_g^{(n)}(x,y) \leq C_0(g(w) + g(v)) \leq C D_g(x,y).$$

This completes the proof of the lemma for the nested fractals.

To prove the case for the GSC, we need some new notations. Let $p_1 = (0, 0, \ldots, 0)$, $p_2 = (1, 0, \ldots, 0)$. Let $L = \{0\} \times [0,1]^{d-1}$ and $R = \{1\} \times [0,1]^{d-1}$ be the left and right face of H_0. Define

$$D_g^{(n)}(L,R) = \min\{g(\gamma) : \gamma \text{ is an } n\text{-chain between } L \text{ and } R\},$$

where an n-chain $\gamma = (w(1),\dots,w(m))$ is called an n-chain between L and R if $K_{w(1)} \cap L \neq \emptyset$ and $K_{w(m)} \cap R \neq \emptyset$. We need another lemma.

Sublemma. *For $\boldsymbol{a} \in (0,1)^k$, $\sup_{n\geq 0} D_g^{(n)}(L,R) < \infty \;\Leftrightarrow\; \sup_{n\geq 0} D_g^{(n)}(p_1,p_2) < \infty$.*

Since the proof of this sublemma uses more notions and another technique, and is quite long, we will prove it in the Appendix in order not to distract the main proof.

For the first assertion, using the same argument again as in the previous proof of the nested fractals, we can also see that $\sup_{n\geq 0} D_g^{(n)}(p_1,p_2) < \infty$ implies that $\sup_{n\geq 0} D_g^{(n)}(x,y) < \infty$ for all $x,y \in K$ (the x_i we choose is an arbitrary vertex of the cube $F_{w(i)}(H_0)$, and note that x_i, x_{i+1} can be connected by a finite number of pairs that can be expressed as affine combination of p_1,p_2 under some F_w and $\sigma \in G$). Thus, it suffices to show that $\boldsymbol{a} \in S$ implies $\sup_{n\geq 0} D_g^{(n)}(p_1,p_2) < \infty$. Suppose otherwise, then by sublemma,

$$\sup_{n\geq 0} D_g^{(n)}(L,R) = \infty.$$

From this, we see that for any given $C_0 > 1$, there exists N_0 large enough such that $D_g^{(N_0)}(L,R) \geq C_0$, i.e., for any two points $x \in L$ and $y \in R$, $D_g^{(N_0)}(x,y) \geq C_0$.

Note that by symmetry, $D_g^{(N_0)}(L,R) = D_g^{(N_0)}(L',R')$ for any two opposite faces L',R' of the unit cube. We will pick some large C_0 (specified later) and show that

$$D_g^{(mN_0)}(p_1,p_2) \geq 2^m. \tag{4.6}$$

Let $m \geq 2$ and $\gamma^{(m)}$ be an mN_0-chain between p_1 and p_2. We decompose $\gamma^{(m)} = \{w(1), w(2),\dots,w(j)\}$ into a sequence of sub-chains as follows. Let $w(1)^-$ be the truncation of $w(1)$ to word length $(m-1)N_0$, and let

$$\mathcal{N}_1 = \cup\{K_v : \; |v| = (m-1)N_0, K_{w(1)^-} \cap K_v \neq \emptyset\}$$

be the neighborhood of $K_{w(1)^-}$ which is contained in a cube having the size three times as $K_{w(1)^-}$. Set $t_0 = 1$ and define

$$t_1 = \inf\{\, t : \; 1 \leq t \leq j, K_{w(t)} \not\subseteq \mathcal{N}_1\},$$

and let $\gamma_1^{(m)} = \{w(1), w(2),\dots,w(t_1-1)\}$. Follow this "first exit time" technique, we can define $\mathcal{N}_i$ and $\gamma_i^{(m)} := \{w(t_{i-1}),\dots,w(t_i-1)\}$, $1 \leq i \leq s$ as the above. Then $\gamma^{(m)}$ is decomposed into subchains $\{\gamma_i^{(m)}\}_{i=1}^s$.

We first consider $\gamma_1^{(m)}$. Note that $H_{w(1)^-} := F_{w(1)^-}(H_0)$ is the subcube containing $K_{w(1)^-}$. Consider a subchain $\gamma_1'^{(m)}$ of $\gamma_1^{(m)}$ such that for $w(i) \in \gamma_1'^{(m)}$, $K_{w(i)^-} \subset \tilde{\mathcal{N}}_1 := \mathcal{N}_1 \setminus H^o_{w(1)^-}$; it corresponds to a chain of mN_0-cells starts from an $(m-1)N_0$-cell T_0, which touches the boundary $\partial H_{w(1)^-}$, and reaches the outer border of $\tilde{\mathcal{N}}_1$ with an exit face contained in a $(d-1)$-dimensional hyperplane H. We can take the union of translates of T_0 towards H and obtain a straight tube-like set T composed of

$(m-1)N_0$-cells. We then reflect the mN_0-cells $K_{w(i)}$ of $\gamma_1'^{(m)}$ (with respect to faces of the $(m-1)N_0$-cells in $\tilde{\mathcal{N}}_1$) towards T successively (like rolling up a carpet). Eventually, we can find a new chain $\tilde{\gamma}_1^{(m)}$ contained in T such that its mN_0-cells cross over two opposite faces of an $(m-1)N_0$-cell $K_{\tilde{w}}$ in T. Note that the number of reflections needed is uniformly bounded by some $k > 0$ which depends only on the dimension d. By using self-similarity and Lemma 2.3, we have

$$g(\gamma_1^{(m)}) \geq g(\gamma_1'^{(m)}) \geq (\frac{a_*}{a^*})^k g(\tilde{\gamma}_1^{(m)}) \geq c_1 C_0 g(w(1)^-). \tag{4.7}$$

We now adjust $\{w(1)^-, w(2)^-, \ldots, w(t_1-1)^-\}$ to a simple $(m-1)N_0$-chain η_1 starting at $w(1)^-$ and ending at $w(t_1-1)^-$, then $|\eta_1| \leq 3^d$. From Lemma 2.3, $g(w(1)^-) \geq c_2 g(\eta_1)$, where $c_2 = 3^{-d}(a_*/a^*)^{3^d}$. Thus using (4.7) , we have

$$g(\gamma_1^{(m)}) \geq c_1 c_2 C_0 g(\eta_1).$$

The same estimate holds for $\gamma_i^{(m)}, 2 \leq i \leq s$, we obtain $g(\gamma_i^{(m)}) \geq c_1 c_2 C_0 g(\eta_i)$.

Next we observe that the concatenation of $\{\eta_1, \cdots, \eta_s\}$ is an $(m-1)N_0$-chain connecting p_1 and p_2, hence $\sum_{i=1}^s g(\eta_i) \geq D_g^{((m-1)N_0)}(p_1, p_2)$. Therefore, we have

$$g(\gamma^{(m)}) = \sum_{i=1}^s g(\gamma_i^{(m)}) \geq c_1 c_2 C_0 \sum_{i=1}^s g(\eta_i) \geq c_1 c_2 C_0 D_g^{((m-1)N_0)}(p_1, p_2).$$

Pick C_0 such that $C_0 \geq 2 + 2c_1^{-1}c_2^{-1}$. Since $\gamma^{(m)}$ is an arbitrary mN_0-chain between p_1 and p_2, we have

$$D_g^{(mN_0)}(p_1, p_2) \geq 2D_g^{((m-1)N_0)}(p_1, p_2) \geq 2^{m-1}C_0 \geq 2^m.$$

This proves (4.6).

Finally we use $\boldsymbol{a}(\varepsilon) = (a_1^{1+\varepsilon}, \ldots, a_k^{1+\varepsilon})$ as in the proof of the nested fractal case to obtain a contradiction, hence $\sup_{n\geq 0} D_g^{(n)}(p_1, p_2) < \infty$.

The proof of the second part of the lemma is similar to the case of nested fractals. □

Theorem 4.4. *Let K be an elementary fractal, and let $\boldsymbol{a} \in \mathcal{M}$. Then $D_{g_{\boldsymbol{a}}}$ is a metric satisfying the MCC if and only if $\boldsymbol{a} \in S$.*

Proof. The necessity follows from Proposition 4.2. To prove the sufficiency, we let $x, y \in K$, and let $\boldsymbol{a} \in S$, and write D_g for $D_{g_{\boldsymbol{a}}}$ for simplicity. Fix any integer $M > 2$, pick $n > n_0$ and so that $a^{*n} \leq \frac{D_g(x,y)}{M}$. Consider an n-chain $\gamma = (w(1), \ldots, w(m))$ that attains $D_g^{(n)}$. Then by (4.4),

$$\sum\nolimits_{i=1}^m g(w(i)) = D_g^{(n)}(x, y) \leq CD_g(x, y). \tag{4.8}$$

We make a decomposition of γ by a "first exit time" technique. Let $s_0 = 0$, define

$$s_1 := \inf\Big\{j : \ \sum\nolimits_{i=1}^j g(w(i)) \geq CM^{-1}D_g(x, y), \ 1 \leq j \leq m\Big\},$$

where C is the same as in (4.8). Inductively, for $t \geq 1$, define

$$s_{t+1} := \inf\Big\{j : \sum_{i=s_t+1}^{j} g(w(i)) \geq CM^{-1}D_g(x,y),\ s_t+1 \leq j \leq m\Big\}.$$

Let $\bar{t}$ be the first integer that $s_{\bar{t}}$ can not be defined, and we assign $s_{\bar{t}} = m$ by convention. Then we have $\bar{t} - 1 \leq M$ (for otherwise, from the construction, $\sum_{i=1}^{m} g(w(i)) \geq (\bar{t}-1)CM^{-1}D_g(x,y) > CD_g(x,y)$), contradicting (4.8)). Also for each $0 \leq t \leq \bar{t} - 1$,

$$\sum_{i=s_t+1}^{s_{t+1}} g(w(i)) \leq CM^{-1}D_g(x,y) + a^{*n} \leq (C+1)M^{-1}D_g(x,y).$$

Now for $1 \leq t \leq \bar{t} - 1$, pick each point $z_t \in K_{w(s_t+1)}$, together with $x = z_0$ and $y = z_{\bar{t}}$, we have by definition of D_g,

$$D_g(z_t, z_{t+1}) \leq \sum_{i=s_t+1}^{s_{t+1}} g(w(i)) + a^{*n} \leq (C+2)M^{-1}D_g(x,y),$$

where C is independent of x, y, M. Since $\bar{t} \leq M+1$, we obtain the MCC of D_g. □

5. Nested fractals

In this section, we give a description of $\mathcal{M}$ for the class of nested fractals. The main idea of representing the recursive weight transfer into matrix is from [26].

Let $\ell_1, \ell_2, \dots, \ell_r$ be such that $0 < \ell_1 < \ell_2 < \cdots < \ell_r$ and $\{\ell_1, \dots, \ell_r\} := \{|x-y| : x, y \in P_0,\ x \neq y\}$. For $n \geq 1$, denote by $P_n = \bigcup_{w \in \Sigma^n} F_w(P_0)$. For each $x \in P_n$ with $n \geq 0$ and for $i = 1, \dots, r$, let $N_n^i(x)$ be the set of all $y \in P_n$ belonging to the same n-cell of x and $|x-y| = \varrho^{-n}\ell_i$; for $y \in N_n^i(x)$, we call the one step move from x to y an *n-move* of type $\langle i \rangle$, $1 \leq i \leq r$. A sequence $x_0, \dots, x_m \in P_n$ is called an *n-walk* if x_j and x_{j+1} are joined in the same n-cell for all $0 \leq j \leq m-1$.

Next we fix a symmetric self-similar weight function g generated by some $\boldsymbol{a} = (a_1, \dots, a_k) \in (0,1)^k$. Pick any $x \in P_0$ and $y \in N_0^i(x)$ for some $1 \leq i \leq r$, consider all the 1-walk $x_0, x_1, \dots, x_m$ such that $x_0 = x$ and $x_m = y$ with $x_1, \dots, x_{m-1} \in P_1 \setminus P_0$ which do not pass through the same point twice. Fix such a 1-walk, for $1 \leq j \leq r$, we count all the $\langle j \rangle$-type 1-moves in this walk. For each $\langle j \rangle$-type 1-move, it can be assigned to a unique 1-cell, and we say that this $\langle j \rangle$-type move has weight a_l if the weight of this 1-cell is a_l. Then we sum up all the weights of these $\langle j \rangle$-type moves and denote it by c_j^i. Let $\mathbf{c}^i = (c_1^i, \dots, c_r^i)$ be the weight of the 1-walk. By the symmetric assumption of $g = g_{\boldsymbol{a}}$, it is clear that $\mathbf{c}^i$ does not depend on the choice of $x \in P_0$ and $y \in N_0^i(x)$. Let $\mathcal{S}_{\boldsymbol{a}}^i$ be the set of $\mathbf{c}^i$ for all these finite number of 1-walks. Then $\mathcal{S}_{\boldsymbol{a}}^i$, $1 \leq i \leq r$ is a finite collection of r-dimensional vectors, and each one is a nonnegative linear combination of weights in $\boldsymbol{a} \in (0,1)^k$ with integer coefficients.

Let

$$\mathcal{K}(\boldsymbol{a}) := \{A :\ A \text{ is a } r \times r\text{-matrix} \ni \text{for } 1 \leq i \leq r, (i\text{-th row of } A) \in \mathcal{S}_{\boldsymbol{a}}^i\}.$$

We call A a *weight transfer matrix*. For $A \in \mathcal{K}(\boldsymbol{a})$, A has nonnegative entries and each row is nonzero. Let λ_A be the largest positive eigenvalue of A. Then it is clear that λ_A is a solution of some polynomial with coefficients generated by $\{a_1, \dots, a_k\}$.

Theorem 5.1. *For a nested fractal, we have*

$$\mathcal{M} = \{\boldsymbol{a} = (a_1, \ldots, a_k) \in (0,1)^k : \ \lambda_A \geq 1 \ \textit{for all} \ A \in \mathcal{K}(\boldsymbol{a})\},$$

and the boundary $S = \{\boldsymbol{a} \in \mathcal{M} : \ \textit{there exists}\ A \in \mathcal{K}(\boldsymbol{a}) \ \textit{such that} \ \lambda_A = 1\}$.

Proof. We first show that if $\boldsymbol{a} = (a_1, \ldots, a_k) \in (0,1)^k$ is such that $\lambda_A < 1$ for some $A \in \mathcal{K}(\boldsymbol{a})$, then $D_{g_{\boldsymbol{a}}}$ is not a metric. The idea is that we use this A to recursively construct n-chains $\{\gamma_n\}_{n\geq 0}$ between two distinct points $x, y \in P_0$ such that $\lim_{n\to\infty} g(\gamma_n) \to 0$. By assumption, for $1 \leq i \leq r$, the i-th row of A is determined by a 1-walk ξ_i between some pair $x, y \in P_0$ with $|x - y| = \ell_i$. We pick $x, y \in P_0$ such that $|x - y| = \ell_1$ and let $\eta_1 = \xi_1$ be the 1-walk between x and y. Let γ_1 be the associated 1-chain of η_1. For $n \geq 1$, we define recursively an n-walk η_n between x, y and denote by γ_n the associated n-chain. Define η_2 by replacing each $\langle i\rangle$-type 1- move in η_1 by the 2-walk $F_w(\sigma(\xi_i))$ for some $\sigma \in G$, where w is the assigned 1-cell of the 1-move. Recursively, we define the n-walk η_n from η_{n-1} in a similar manner. For the weight $\mathbf{c}^1$ of η_1 and $\mathbf{1} = (1, \ldots, 1)$, we have

$$g(\gamma_n) = \mathbf{c}^1 A^{n-1} \mathbf{1}^t \to 0 \quad \text{ as } n \to \infty,$$

by $\lambda_A < 1$. This implies that $D_g(x, y) = 0$ and consequently D_g is not a metric.

Conversely, We use contradiction to show that if $\lambda_A \geq 1$ for all $A \in \mathcal{K}(\boldsymbol{a})$, then $D_{g_{\boldsymbol{a}}}$ is a metric on K. We fix $\boldsymbol{a}$ and define the operator $G_{\boldsymbol{a}} : [0,\infty)^r \to [0,\infty)^r$ by

$$(G_{\boldsymbol{a}}(\boldsymbol{x}))_i = \min_{\mathbf{c}^i \in \mathcal{S}_a^i} \Big\{ \sum_{j=1}^{r} c_j^i x_j \Big\} = \min_{\mathbf{c}^i \in \mathcal{S}_a^i} \{\langle \mathbf{c}^i, \boldsymbol{x}^t\rangle\}, \qquad 1 \leq i \leq r,$$

where $\boldsymbol{x} = (x_1, \ldots, x_r)$. Note that for each $1 \leq i \leq r$, there exists $\mathbf{c}^i \in \mathcal{S}_{\boldsymbol{a}}^i$, such that $(G_{\boldsymbol{a}}(\boldsymbol{x}))_i = \langle \mathbf{c}^i, \boldsymbol{x}^t\rangle$. This defines a matrix $A_{\min} \in \mathcal{K}(\boldsymbol{a})$ (depends on $\boldsymbol{x}$) such that $G_{\boldsymbol{a}}(\boldsymbol{x}) = A_{\min}\boldsymbol{x}^t$.

By using the same technique as in [26, Lemma 3.3], we have $G_{\boldsymbol{a}}(B) \subseteq B$ for $B = \{\boldsymbol{x} \in \mathbb{R}^r : \ 0 \leq x_1 \leq x_2 \leq \cdots \leq x_r\}$. For completeness, we provide a proof for this with a slight modification.

Fix $p \in P_0$, $q_i \in N_0^i(p)$ and $q_i' \in N_0^{i-1}(p)$ for $2 \leq i \leq r$. Let $R_{q_i q_i'}$ be the reflection between q_i and q_i'. Let $V_i = \{z \in \mathbb{R}^n : |z - q_i'| \leq |z - q_i|\}$ and $\bar{z} = R_{q_i q_i'} z$. Let $T_i : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$T_i(z) = \begin{cases} z, & z \in V_i, \\ \bar{z}, & \text{otherwise}. \end{cases}$$

Given $\boldsymbol{x} \in B$ and $i \geq 2$, consider $\mathbf{c}^i \in A_{\min}$, then there is a 1-walk ξ_i between p and q_i with weight $\mathbf{c}^i$. Express ξ_i by $p = x_0, x_1, \ldots, x_m = q_i$ into 1-moves $\{(x_j, x_{j+1})\}_{j=0}^{m-1}$. Then we see that $T_i(\xi_i)$ is a 1-walk between p and q_i', and $(T_i(x_j), T_i(x_{j+1}))$ is a 1-move in the same cell as (x_j, x_{j+1}), which has type smaller than or equal to (x_j, x_{j+1}) because $|T_i(x_j) - T_i(x_{j+1})| \leq |x_j - x_{j+1}|$. Denote by $\mathbf{t} = (t_1, \ldots, t_r)$ the weight of the 1-walk $T_i(\xi_i)$. Then we have

$$(G_{\boldsymbol{a}}(\boldsymbol{x}))_{i-1} \leq \langle \mathbf{t}, \mathbf{x}^t\rangle \leq \langle \mathbf{c}^i, \mathbf{x}^t\rangle = (G_{\boldsymbol{a}}(\boldsymbol{x}))_i,$$

since $\boldsymbol{x} \in B$. This proves $G_{\boldsymbol{a}}(B) \subseteq B$.

Consider the normalization $\widetilde{G}(\boldsymbol{x}) = G(\boldsymbol{x})/\sum_i G(\boldsymbol{x})_i$ on $B_\varepsilon = \{x \in B : \sum_i x_i,\ x_1 \geq \varepsilon\}$. Then $\widetilde{G}(\boldsymbol{x})(B_\varepsilon) \subset B_\varepsilon$. By using the Brouwer fixed point theorem, there is a fixed point $\widetilde{G}(\boldsymbol{x}) = \boldsymbol{x}$. It follows that $G(\boldsymbol{x}) = \lambda \boldsymbol{x} = A_{\min}\boldsymbol{x}$ where $\lambda = \sum_i G(\boldsymbol{x})_i$, and is the maximum eigenvalue of $A_{\min}$ (for detail, see [26, Proposition 3.4]).

Finally, for $n \geq 0$, let $\mathbf{z}_n = (z_{n,1}, \dots, z_{n,r})$ be a vector of positive real numbers such that $z_{n,i} = D_g^{(n)}(p, q)$, where $p, q \in P_0$ and $|p - q| = \ell_i$, $1 \leq i \leq r$. Then

$$\mathbf{z}_n = G_{\boldsymbol{a}}(\mathbf{z}_{n-1}), \qquad n \geq 1.$$

Denote $C' = \min\{x_i^{-1} : 1 \leq i \leq r\}$. Then $\mathbf{z}_0 = (1, \dots, 1) \geq C'\boldsymbol{x}$ so that $\mathbf{z}_1 \geq G_{\boldsymbol{a}}(C'\boldsymbol{x}) = C'\lambda\boldsymbol{x}$. In general, we have $\mathbf{z}_n \geq G_{\boldsymbol{a}}(C'\lambda^{n-1}\boldsymbol{x}) = C'\lambda^n\boldsymbol{x}$ for all $n \geq 1$. On the other hand, if $D_{g_{\boldsymbol{a}}}$ is not a metric on K, then by (4.2), $\lim_{n\to\infty} D_{g_{\boldsymbol{a}}}^{(n)}(x, y) = 0$ for any $x, y \in P_0$. This contradicts the fact that $\lambda \geq 1$ and $\boldsymbol{x}$ is positive. Thus $\boldsymbol{a} \in \mathcal{M}$ and the proof of the first assertion is complete.

To prove the second assertion, we denote by $S' := \{\, \boldsymbol{a} \in \mathcal{M} : \ \exists\, A \in \mathcal{K}(\boldsymbol{a}) \ni \lambda_A = 1\}$. For any $\boldsymbol{a} = (a_1, \dots, a_k) \in S'$, we may assume $A(\boldsymbol{a}) \in \mathcal{K}(\boldsymbol{a})$ such that $\lambda_{A(\boldsymbol{a})} = 1$. For any $\delta > 1$, denote by $\boldsymbol{a}^\delta = (a_1^\delta, \dots, a_k^\delta)$, each nonzero entry of $A(\boldsymbol{a}^\delta)$ is strictly smaller than that of $A(\boldsymbol{a})$. Denote by

$$c_0 = \max\{\, c > 1 : \text{ each entry of } cA(\boldsymbol{a}^\delta) \text{ is smaller than or equal to that of } A(\boldsymbol{a})\,\}.$$

Thus $c_0 > 1$ and by using the Perron-Frobenius theorem, we have

$$c_0\lambda_{A(\boldsymbol{a}^\delta)} = \lambda_{c_0A(\boldsymbol{a}^\delta)} \leq \lambda_{A(\boldsymbol{a})} = 1,$$

and hence $\lambda_{A(\boldsymbol{a}^\delta)} \leq c_0^{-1} < 1$, which implies that $\boldsymbol{a}^\delta \in \mathcal{M}^c$ for any $\delta > 1$. Hence $\boldsymbol{a} \in S$, which implies $S' \subseteq S$. On the other hand, for any $\boldsymbol{a} = (a_1, \dots, a_k) \in S$, we show that $\boldsymbol{a} \in S'$. If $\boldsymbol{a} \notin S'$, then $\lambda_{A(\boldsymbol{a})} > 1$ for all $A(\boldsymbol{a}) \in \mathcal{K}(\boldsymbol{a})$. It is clear that $\lambda_{A(\boldsymbol{a})}$ is continuous with respect to $\boldsymbol{a}$ and the cardinality of $\mathcal{K}(\boldsymbol{a})$ is finite. Thus for all s close to 1, for all $A(\boldsymbol{a}^s) \in \mathcal{K}(\boldsymbol{a}^s)$, we have $\lambda_{A(\boldsymbol{a}^s)} > 1$. Hence $\boldsymbol{a} \in \mathcal{M}^o$, a contradiction. This shows that $\boldsymbol{a} \in S'$ and $S \subseteq S'$.Thus we have $S = S'$. □

Remark. The size of the families $\mathcal{S}_{\boldsymbol{a}}^i, 1 \leq i \leq r$, can be cut down substantially for calculation. We define the *essential class* $\tilde{\mathcal{S}}_{\boldsymbol{a}}^i, 1 \leq i \leq r$ to be the set of $\mathbf{c} := (c_1, \dots, c_r) \in \mathcal{S}_{\boldsymbol{a}}^i$ that is smallest (in the sense of coordinatewise ordering) for $\mathcal{S}_{\boldsymbol{a}}^i$. This is because for $\mathbf{c}' \in \mathcal{S}_{\boldsymbol{a}}^i$ with $\mathbf{c} \leq \mathbf{c}'$, if we replace the row in the weight transfer matrix A containing $\mathbf{c}$ by $\mathbf{c}'$, then the maximal eigenvalue increases.

Let $\tilde{\mathcal{K}}(\boldsymbol{a})$ be the $r \times r$ matrices A such that the i-th row is in $\tilde{\mathcal{S}}_{\boldsymbol{a}}^i$, then clearly

$$\mathcal{M} = \{\boldsymbol{a} = (a_1, \dots, a_k) \in (0, 1)^k : \ \lambda_A \geq 1 \text{ for all } A \in \tilde{\mathcal{K}}(\boldsymbol{a})\}.$$

In the following, we use the *Lindstrøm snowflake* to give an illustration of the theorem and the remark.

Let $p_i = (\cos(i\pi/3), \sin(i\pi/3))$, $i = 1, \dots, 6$ and $p_7 = (0, 0)$. Define $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ by $F_i(x) = (x - p_i)/3 + p_i$ for $i = 1, \dots, 7$. The Lindstrøm snowflake K is the self-similar set generated by the IFS $\{F_i\}_{i=1}^7$ (see Figure 3). It has boundary $V_0 = \{p_1, \dots, p_6\}$.

In this case, $\Sigma = \{1, \ldots, 7\}$. We consider a symmetric self-similar weight function $g = g_{a,b}$ on Σ^* defined by

$$g(i) = \begin{cases} a, & 1 \le i \le 6, \\ b, & i = 7, \end{cases} \tag{5.1}$$

where $a, b \in (0, 1)$. (See Figure 3.)

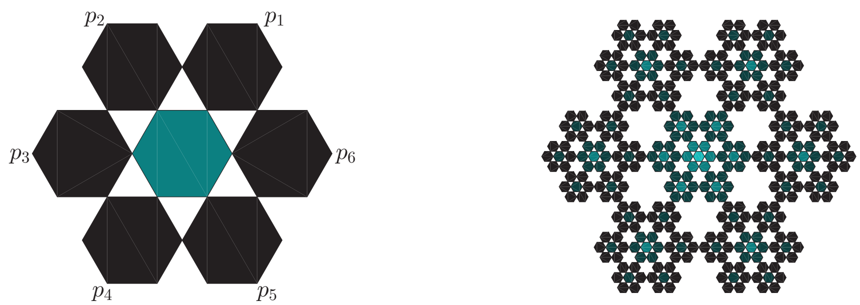

FIGURE 3. Lindstrøm snowflake, and the weight function $g_{a,b}(w)$ for $|w| = 1, 3$.

Corollary 5.2. *Let $g = g_{a,b}$ be defined as the above. Then $D_{g_{a,b}}$ is a metric if and only if $3a \ge 1$ and $2a + b \ge 1$. Moreover, D_g satisfies the MCC if and only if $(a, b) \in \{3a = 1, b \ge \frac{1}{3}\} \cup \{2a + b = 1, b \le \frac{1}{3}\}$.*

Proof. The second conclusion is a consequence of the first part and Theorem 4.4.

To prove the first part, we adopt the notations and setup preceding Theorem 5.1. It is easy to see that the Lindstrøm snowflake has three types of 1-move, that is $\ell_1 = |\overline{p_1 p_2}|$, $\ell_2 = |\overline{p_1 p_3}|$ and $\ell_3 = |\overline{p_1 p_4}|$. Let $\boldsymbol{a} = (a, b)$ be as in (5.1), and consider $p_2 \in N_0^1(p_1)$. For a 1-walk of p_1 to p_2 in $P_1 \setminus P_0$, by using the above remark and elementarily checking case by case, we obtain two vectors in the essential classes $\tilde{\mathcal{S}}_{\boldsymbol{a}}^1$. Similarly, we can calculate $\tilde{\mathcal{S}}_{\boldsymbol{a}}^2$ and $\tilde{\mathcal{S}}_{\boldsymbol{a}}^3$ (see Figure 4).

$\tilde{\mathcal{S}}_{\boldsymbol{a}}^1 = \{(0, 2a, 0), (b, 0, 2a)\}$;

$\tilde{\mathcal{S}}_{\boldsymbol{a}}^2 = \{(a + b, a, a), (0, b, 2a), (a, a, a + b), (0, 3a, 0)\}$;

$\tilde{\mathcal{S}}_{\boldsymbol{a}}^3 = \{(0, 4a, 0), (2a + b, 2a, 0), (a + b, 2a, a), (a, a + b, a), (0, 0, 2a + b)\}$.

The $A \in \tilde{\mathcal{K}}(\boldsymbol{a})$ are formed by picking one vector in each of the $\tilde{\mathcal{S}}_{\boldsymbol{a}}^i$. It can be checked directly that $\lambda_A \ge 1$ for all $A \in \tilde{\mathcal{K}}(\boldsymbol{a})$ is equivalent to

$$\begin{cases} 3a \ge 1, \\ 2a + b \ge 1. \end{cases} \tag{5.2}$$

In particular, the two determining matrices for $\mathcal{M}$ and S are

$$A_1 = \begin{pmatrix} 0 & 2a & 0 \\ 0 & 3a & 0 \\ 0 & 4a & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 2a & 0 \\ 0 & b & 2a \\ 0 & 0 & 2a + b \end{pmatrix},$$

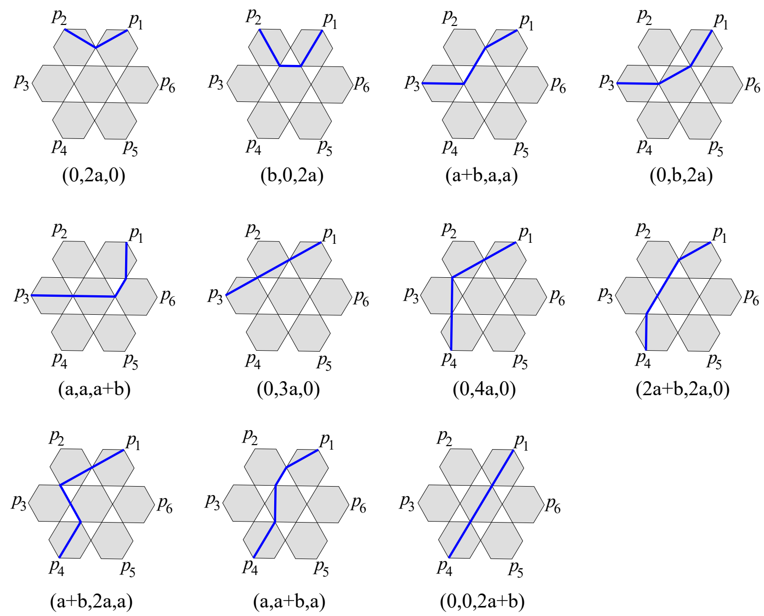

FIGURE 4. The essential classes $\tilde{\mathcal{S}}_{\boldsymbol{a}}^1$ (two elements), $\tilde{\mathcal{S}}_{\boldsymbol{a}}^2$ (four elements), $\tilde{\mathcal{S}}_{\boldsymbol{a}}^3$ (five elements)

with the two spectral radii $\lambda_{A_1} = 3a$ and $\lambda_{A_2} = 2a + b$ respectively. □

Remark. In fact, it is easy to see that the two conditions in (5.2) are necessary for D_g to be a metric by looking at the chains along the straight lines contained in the fractal. For example, the line $\overline{p_1p_3}$ gives $3a \geq 1$ and the line $\overline{p_1p_4}$ gives $2a + b \geq 1$.

6. Sierpinski carpet

In this section, we consider the standard Sierpinski carpet GSC$(2, 3, Q')$, where $Q' = Q \setminus \{[1/3, 2/3]^2\}$. Let $F_i(x) = (x + 2p_i)/3, i = 1, \ldots, 8$, be contractive maps on $\mathbb{R}^2$, with the p_i specified as in Figure 5.

For any $(a, b) \in (0, 1)^2$, define $g := g_{a,b} : \Sigma^* \to (0, 1]$ by

$$g(i) = \begin{cases} a, & \text{if } i \in \{1, 3, 5, 7\}, \\ b, & \text{if } i \in \{2, 4, 6, 8\}, \end{cases} \tag{6.1}$$

see Figure 5 for the case $|w| = 3$.

Let $\Pi = \{(a, b) \in (0, 1)^2 : \ 2a + b \geq 1 \text{ and } a + 2b \geq 1\}$, and let $\partial\Pi := \{(a, b) \in \Pi : \ a + 2b = 1 \text{ or } 2b + a = 1\}$ be the boundary of Π in $(0, 1)^2$.

Theorem 6.1. *$D_{g_{a,b}}$ is a metric on K if and only if $(a, b) \in \Pi$. Consequently, $D_{g_{a,b}}$ satisfies the MCC if and only if $(a, b) \in \partial\Pi$.*

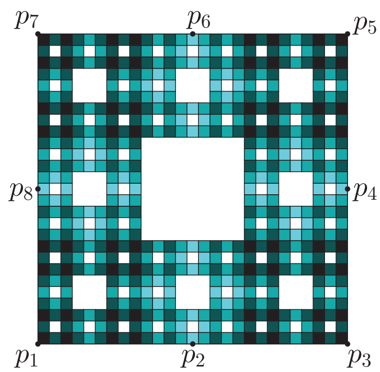

FIGURE 5. Sierpinski carpet, and the weight function $g_{a,b}(w)$ for $|w| = 3$.

The MCC follows directly from the first part and Theorem 4.4. The necessity that $D_{g_{a,b}}$ is a metric is due to Kigami. It is rather straightforward. We provide a proof in the following for completeness.

Proof of the necessity. For $n \geq 1$, let γ_n be a chain of n-cells connecting p_1 and p_3 and each cell intersects the line $\overline{p_1 p_3}$ (i.e., $\gamma_1 = \{1, 2, 3\}$, $\gamma_2 = \{11, 12, 13; 21, 22, 23; 31, 32, 33\}$, and so on). By elementary calculations, we obtain

$$g(\gamma_n) = (2a + b)^n.$$

Since D_g is a metric, $\inf_{n \in \mathbb{Z}^+} (2a + b)^n \geq D_g(p_1, p_3) > 0$ so that $2a + b \geq 1$.

Next, we consider the chains $\{\gamma'_n\}_{n\geq 1}$ of n-cells connecting p_2 and p_4, and each cell intersects the line segment $\overline{p_2 p_4}$ (i.e., $\gamma'_1 = \{2, 3, 4\}$, $\gamma'_2 = \{22, 23, 24; 38, 37, 36; 42, 43, 44\}$, and so on). Then

$$g(\gamma'_n) = (a + 2b)^n.$$

Same as the above, we have $a + 2b \geq 1$. □

The proof of the sufficiency of the theorem is more complicate. The following corollary is a simple consequence of Lemma 2.3 adapted to the situation we need.

Corollary 6.2. *Suppose $K_u \sim K_v$ and $|u| = |v|$. Then*

$$\frac{g(u)}{g(v)} = \frac{a}{b} \ \ or \ \ \frac{b}{a}.$$

Furthermore, if γ is a chain contained in K_u, if we reflect it along the intersection line L, and denote the reflected chain by γ', then γ' is in K_v, and $\frac{g(\gamma')}{g(\gamma)} = \frac{a}{b}$ or $\frac{b}{a}$.

In view of Theorem 3.1 and (4.2), we only need to consider the n-chains between arbitrary two fixed distinct points in K. As in the proof, we will use Corollary 6.2 frequently for reflection. We need a few more lemmas.

The following lemma is easy in view of the proof of necessity.

Lemma 6.3. *Let $(a, b) \in \Pi$ and let $n \geq 0$. Suppose γ is an n-chain satisfying either*
(i) γ connects p_1 and p_3 and all its cells intersect the line segment $\overline{p_1 p_3}$; or
(ii) γ connects p_2 and p_8 and all its cells intersect the line segment $\overline{p_2 p_8}$.
Then

$$g(\gamma) \geq 1.$$

We separate the proof into two cases. Define

$$\Pi_1 = \{(a, b) \in \Pi : a \leq b\} \quad \text{and} \quad \Pi_2 = \{(a, b) \in \Pi : a \geq b\}.$$

The case for Π_1. For $n \geq 1$, we let $P_n = F_{1^{n-1}}(0, 1/2)$ and $P'_n = F_{1^{n-1}}(0, 1/3)$, and use L_n and L'_n to denote the lines $y = \frac{1}{2\cdot 3^{n-1}}$ and $y = \frac{1}{3^n}$, respectively. Then L_n passes through P_n, and L'_n passes through P'_n. We set $D'_0 = [0, 1]^2$, and denote by D_n (or D'_n) the rectangle enclosed by the lines $x = 0, 1$, $y = 0$, and $y = L_n$ (or $y = L'_n$ respectively) (see Figure 6).

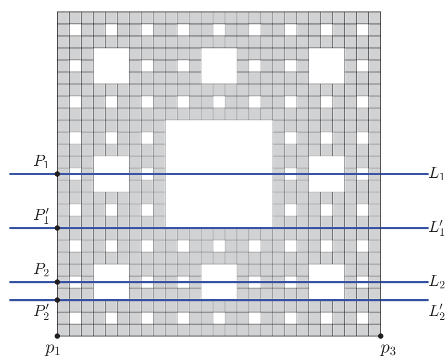

FIGURE 6. Lines for reflection in the case Π_1.

For a cell K_w, we define its *center* to be $F_w(1/2, 1/2)$. We also define an (L_n)-reflection to be reflecting a cell along the line L_n, and similarly an (L'_n)-reflection.

Lemma 6.4. *Suppose $(a, b) \in \Pi_1$. Let K_w be a cell with $|w| \geq n$.*
(i) If K_w has center in $D'_{n-1} \setminus D_n$, and let K_u be the reflected cell of K_w along L_n, then K_u is centered in D_n and $g(u) \leq g(w)$.
(ii) If K_w has center in $D_n \setminus D'_n$, and let K_u be the reflected cell of K_w along L'_n, then K_u is centered in D'_n and $g(u) \leq g(w)$.

Proof. In the first case, the center of $K_{w|_{n-1}}$ is on L_n, and the (L_n)-reflection sends $K_{w|_{n-1}}$ to itself. It follows that K_u is centered in D_n, and $g(u) = g(w)$ by symmetry. In the second case, one of the line segment of $\partial K_{w|_n}$ is on L'_n, hence the (L'_n)-reflected cell is in D'_n, and $g(u) = (a/b) \cdot g(w) \leq g(w)$ by Corollary 6.2. $\square$

Lemma 6.5. *Suppose $(a,b) \in \Pi_1$. For $n \geq 0$, let γ be an n-chain contained in K connecting p_1 and p_3. Then there exists another n-chain $\widetilde{\gamma}$ connecting p_1 and p_3, with $\overline{p_1p_3} \subset \cup\widetilde{\gamma}$, and*

$$g(\widetilde{\gamma}) \leq g(\gamma).$$

Proof. Denote $\gamma_1 := \gamma$, the given chain contained in K connecting p_1 and p_3.

For any cell K_w in γ_1, K_w has center in D'_0. We then apply the (L_1)-reflection to those cells centered in $D'_0 \setminus D_1$ to obtain another chain that the cells are centered in D_1 (by Lemma 6.4). We do the (L'_1)-reflection for those cells in the new chain centered in $D_1 \setminus D'_1$ to obtain another chain, denoted by γ_2. Then the cells K_w of γ_2 are centered in D'_1.

Inductively we apply this procedure for n times with (L_i)-reflection and (L'_i)-reflection for $i = 1, \ldots, n$. Then we obtain a chain γ_{n+1} such that each K_w in γ_{n+1} has center in D'_n. By Lemma 6.4,

$$g(\gamma_{n+1}) \leq \cdots \leq g(\gamma_2) \leq g(\gamma_1).$$

Denote γ_{n+1} by $\widetilde{\gamma}$. Now all cells K_u in $\widetilde{\gamma}$ have word length $|u| = n$ and center in D'_n. Then one of the line segment of ∂K_u is on $\overline{p_1p_3}$. Note that all the reflections keep p_1, p_3 in the two end cells. This yields $\overline{p_1p_3} \subset \cup\widetilde{\gamma}$. Since

$$g(\widetilde{\gamma}) \leq g(\gamma_1),$$

the proof is completed. □

Proof of "sufficiency" for Π_1. Let $(a,b) \in \Pi_1$, then $2a + b \geq 1$ and $a \leq b$. Assume that D_g is not a metric. Then by Theorem 3.1 and (4.2), for each $n \geq 1$, there exists an n-chain γ_n connecting p_1 and p_3 such that $\lim_{n\to\infty} g(\gamma_n) = 0$. By Lemma 6.5, we obtain a new chain $\widetilde{\gamma}_n$ such that $\overline{p_1p_3} \subset \cup\widetilde{\gamma}_n$, and

$$g(\widetilde{\gamma}_n) \to 0 \quad \text{as } n \to \infty.$$

On the other hand, by Lemma 6.3 (i), we see that $g(\widetilde{\gamma}_n) \geq 1$ for all $n \geq 1$, a contradiction. Hence D_g is a metric. □

The case for Π_2. The idea of proof is the same as the case Π_1, but the geometry of the reflection is slightly more complicate. We will concentrate on the set

$$\Omega = K_{37} \cup K_{36} \cup K_{35} \cup K_{41} \cup K_{42} \cup K_{43}.$$

For $n \geq 1$, we denote $q_n = F_{36^{n-1}}(1/2, 1/2)$ and $q'_n = F_{42^{n-1}}(1/2, 1/2)$. We define ℓ_n and ℓ'_n to be the two parallel lines passing through q_n and q'_n with slope -1. Clearly, whenever ℓ_n (ℓ'_n) intersects the "interior" of a cell K_w, $|w| \geq n$, the center of K_w lies in ℓ_n (ℓ'_n respectively).

Let ℓ_* and ℓ'_* be the lines passing through the points q_1 and q'_1 with the same slope 1. Let M_1 denote the hexagon enclosed by lines $y = 2/9$, $y = 4/9$, ℓ_1, ℓ'_1, ℓ_* and ℓ'_*; for $n \geq 2$, let M_n denote the rectangle enclosed by the lines ℓ_*, ℓ'_*, ℓ_n and ℓ'_n, see Figure 7. Also we let ℓ be the line passing through the point $q = F_3(1/2, 1) = F_4(1/2, 0)$ (in the center of Ω) with slope -1.

The following are some simple geometric properties of the notions we defined.

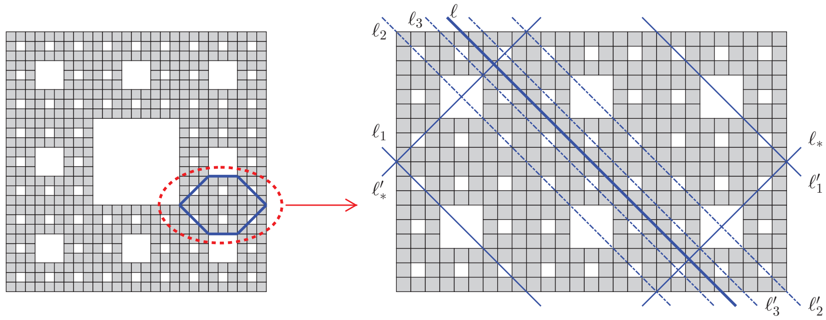

FIGURE 7. Ω and lines for reflection.

(i) Let $d_n = \frac{\sqrt{2}}{2}3^{-n}$. Then $d_n = \text{dist}(\ell_n, \ell'_n) = \text{dist}(\ell_n, \ell_{n-1}) = \text{dist}(\ell'_n, \ell'_{n-1})$. Therefore, ℓ'_n is the reflection of ℓ_{n-1} through ℓ_n, and ℓ_n is the reflection of ℓ'_{n-1} through ℓ'_n.

(ii) $\ell \cap \big(F_3(K) \cup F_4(K)\big)$ is a line segment contained in K, and lies in between ℓ_n and ℓ'_n for all $n \geq 1$. If K_w is an n-cell centered in ℓ_n or ℓ'_n, then by $\text{diam}(K_w) = 2d_n$, K_w intersects ℓ.

(iii) Let O be the center of an n-cell K_w. If $O \in M_n$, then $O \in \ell_n \bigcup \ell'_n$; if $O \in M_{n-1} \setminus M_n$, then $O \in \ell_{n-1} \bigcup \ell'_{n-1}$.

(iv) For K_w with $|w| \geq n$, if K_w has center in M_n, then $K_{w|_n}$ also has center in M_n; if K_w has center in $M_{n-1} \setminus M_n$, then $K_{w|_n}$ has center in $M_{n-1} \setminus M_n$.

Suppose $n \geq 2$, K_w is a cell with $|w| \geq n$ and has center in M_{n-1}. From (iii)-(iv), $K_{w|_n}$ has center in $\ell_{n-1} \cup \ell_n$ (or $\ell'_{n-1} \cup \ell'_n$). We define the (n)-*reflection* to be the reflection of K_w with respect to ℓ_n (or ℓ'_n respectively).

Lemma 6.6. *Let $(a, b) \in \Pi_2$. Suppose K_w with $|w| \geq n \geq 2$ has center in M_{n-1}. Let K_u be the (n)-reflected cell of K_w, then K_u is centered in M_n, and*

$$g(u) \leq g(w).$$

Proof. We assume that $w = w_1 \cdots w_m$ and $u = u_1 \cdots u_m$, where $m \geq n$. The assumption on K_w implies the center of $K_{w|_n}$ also lies in M_{n-1} (by (iv)), and hence lies in $\ell_n \cup \ell'_n$ or $\ell_{n-1} \cup \ell'_{n-1}$ (by (iii)). For the first case, the (n)-reflection of $K_{w|_n}$ is itself, then K_u is also a subcell of $K_{w|_n}$. By the symmetry, we have $r_{u_i} = r_{w_i}$ for all $i \geq n+1$ (the r_j is defined in (6.1)), so that $g(u) = g(w)$.

For the second case, let us assume that the center of $K_{w|_n}$ lies in ℓ_{n-1}, then the center of $K_{w|_{n-1}}$ also lies in ℓ_{n-1}, and $w_n = 3$ or 7, so that $r_{w_n} = a$. It is easy to check that $K_{u|_{n-1}}$ and $K_{w|_{n-1}}$ share the same line segment so that $\frac{g(u|_{n-1})}{g(w|_{n-1})}$ is either $\frac{a}{b}$ or $\frac{b}{a}$. Furthermore $u_n = 2$ if $w_n = 7$, and $u_n = 8$ if $w_n = 3$ so that $r_{u_n} = b$. Thus $\frac{g(u|_n)}{g(w|_n)}$ is either $\frac{b}{a} \cdot \frac{a}{b}$ or $\frac{b}{a} \cdot \frac{b}{a}$ (Corollary 6.2). By using that $a \geq b$, we have $g(u|_n) \leq g(w|_n)$. By symmetry, we have $r_{u_i} = r_{w_i}$ for all $i \geq n+1$, so that $g(u) \leq g(w)$. □

Lemma 6.7. *Suppose $(a, b) \in \Pi_2$. Let $n \geq 2$ and let γ be an n-chain contained in $K_{36} \cup K_{42}$, connecting $p_* = F_{36}(1, 0)$ and $p'_* = F_{42}(0, 1)$. Then there exists another n-chain $\widetilde{\gamma}$ with the two end cells touching ℓ_* and ℓ'_*, all its cells intersecting the line segment $\ell \cap \Omega$, and*

$$g(\widetilde{\gamma}) \leq g(\gamma).$$

Proof. Let $\gamma_1 := \gamma$ be a given n-chain contained in $K_{36} \cup K_{42}$, connecting p_* and p'_*. Then each cell has word length n and has center in M_1.

By applying the (2)-reflection, we obtain another n-chain whose cells have centers in M_2 (by Lemma 6.6). We denote this chain by γ_2. We carry out the operations m-reflection for m from 2 to n inductively, and obtain a chain γ_{n+1}, such that each K_w in γ_{n+1} has center in M_{n+1}.

We denote by $\widetilde{\gamma} = \gamma_{n+1}$ the same as Lemma 6.5 for Π_1 with some obvious adjustments (use (ii) to guarantee the cells in $\widetilde{\gamma}$ intersect ℓ), and arrive the conclusion. □

Proof of the "sufficiency" for Π_2. The proof is the same as for the case Π_1, using Lemma 6.7 and Lemma 6.3 (ii) instead. □

7. Application to time change

In this section we consider the time change and the sub-Gaussian heat kernel estimates by summarizing the techniques in [22, 23, 14]. We show that the admissible metrics D_g defined by weights on S allow us to give a concrete class of geodesic metrics that admit a two-sided sub-Gaussian estimates.

Recall that for two metrics d_1 and d_2 on M, d_1 is said to be quasisymmetric to d_2 if *there exists a homeomorphism h from $[0, \infty)$ to itself with $h(0) = 0$ such that for any $t > 0$ and $x, y, z \in M$, $d_2(x, z) < h(t) d_2(x, y)$ whenever $d_1(x, z) < t d_1(x, y)$* [16]. In [24, Theorem 15.7], Kigami proved the following proposition.

Proposition 7.1. *Let d be the resistance metric on K if K is a nested fractal, or the Euclidean metric if K is a GSC. Let $\boldsymbol{a} \in \mathcal{M}$. Then $D_g := D_{g_{\boldsymbol{a}}}$ is quasisymmetric to d.*

We call a measure μ satisfies the (volume) doubling condition (VD) if there exists $C > 0$ such that $\mu(2B) \leq C\mu(B)$ for any ball B. For a symmetric self-similar measure (i.e., $\mu_i = \mu_{\sigma(i)}$ for $\sigma \in G$ and $i \in \Sigma$), from [22, Theorems 1.6.6 and 3.4.5], we know that μ is volume doubling under d in Proposition 7.1, hence by quasisymmetry, μ is also volume doubling under D_g.

In the following, we consider the time change of the standard Brownian motion on K with respect to the symmetric self-similar measures. Let ρ be the renormalization factor of the associated Dirichlet form in $L^2(K, \mathcal{H}^\alpha)$ (see the paragraph of (1.3) in the Introduction). We define the *capacity* $\mathrm{cap}(A, \Omega)(= R(A, \Omega)^{-1})$ between two open sets A, Ω with $A \Subset \Omega$ by

$$\mathrm{cap}(A, \Omega) := \inf\Big\{\mathcal{E}(u) : \ u \in \mathcal{F}, u|_A = 1, u = 0 \text{ on } \Omega^c\Big\}.$$

Let $B = B(x,r) := \{y \in K : |x-y| < r\}$ be a metric ball in K under the metric d in Proposition 7.1. We use $\mathrm{cap}(B, 2B)$ to denote the capacity of two concentric balls B and $2B$. Then, we have [1, 4]

$$\mathrm{cap}(B, 2B) \asymp r^{\xi}, \tag{7.1}$$

where $\xi = -\log\rho/\log\ell$ for GSC, and $\xi = -1$ for nested fractals. We denote this property by $(\mathrm{cap})_d$.

Next let us consider a symmetric self-similar measure μ with weights $\{\mu_i\}_{i=1}^N$ and $\mu_i\rho < 1$. Let $g := g_{\boldsymbol{a}}$ be the symmetric weight function with

$$\boldsymbol{a} := \boldsymbol{a}(\lambda) = \Big((\mu_1\rho)^\lambda, (\mu_2\rho)^\lambda, \dots, (\mu_N\rho)^\lambda\Big) / \sim_G,$$

where $\lambda > 0$ is such that $\boldsymbol{a} \in \mathcal{M}$, and let D_g be the associated admissible metric. Then D_g is quasisymmetric to d as in Proposition 7.1, hence μ is also a doubling measure with respect to D_g. We use $B_D = B_D(x,r)$ to denote the balls for D_g, and express (7.1) in terms of μ.

By (7.1) and quasisymmetry, we can easily obtain the following capacity estimate. Let B_D be a ball with radius r under D_g, then

$$\mathrm{cap}(B_D, 2B_D) \asymp \frac{\mu(B_D)}{r^{1/\beta}}.$$

We denote this property by $(\mathrm{cap})_D$.

We say that the *elliptic Harnack inequality* holds if there is $C > 0$ such that for any nonnegative harmonic function u on $2B$,

$$\sup_B u \le C \inf_B u,$$

where the ball B is with respect to some reference metric; we denote by $(H)_d$ if the metric balls are under metric d, and $(H)_D$ if the metric balls are under D_g. It is known that for the standard Dirichlet forms constructed on the elementary fractals, condition $(H)_d$ holds. By quasisymmetry and [7, Lemma 5.3], we obtain $(H)_D$.

Outline of the proof of Theorem 1.8. Notice that μ is a volume doubling measure. By [14, Theorem 1.1], we know that under conditions (VD) and (RVD) (reversed VD), we have $(H)_D + (\mathrm{cap})_D \Leftrightarrow (UE) + (NLE)$, (Note that the (RVD) follows from (VD) if K is unbounded, and here we can extend K to infinity by self-similarity.) As the two conditions on the left side are satisfied, the right side also hold. This implies the first part of the theorem.

Since D_g satisfies the MCC if and only if $\lambda = \lambda_0$ such that $\boldsymbol{a}(\lambda_0) \in S \subset \mathcal{M}$ (Theorem 4.4), by a standard chain argument (see [12, p.39-41]), we see that (NLE) implies the off-diagonal lower estimate for such $\boldsymbol{a}$ and D_g with λ_0.

Remark. Note that the renormalization factors for the elementary fractals are $0 < \rho < N$ (see $(*)$ below), we can conclude that each $\boldsymbol{a} \in \mathcal{M}$ can be expressed as in (1.4),

$$\boldsymbol{a} = \boldsymbol{a}(\lambda) := ((\rho\mu_1)^\lambda, \dots, (\rho\mu_k)^\lambda) \quad \text{for some} \quad \lambda > 0,$$

and hence Theorem 1.8 applies to all $\boldsymbol{a}$ in $\mathcal{M}$ and in S. Indeed, with a slight abuse of notation, we write $\boldsymbol{a} = (a_1, \dots, a_N)$ where $a_i = a_j$ for $i \sim_G j$. Then there exists $\lambda > 0$ such that $\sum_{i=1}^N a_i^{1/\lambda} = \rho$ (since the sum goes to 0 as $\lambda \to 0$, and goes to N as $\lambda \to \infty$). Let $\mu_i = a_i^{1/\lambda}/\rho$, it defines a symmetric self-similar measure, and $\boldsymbol{a}$ has the expression as asserted.

$(*)$ For nested fractal, $\rho < 1$; for GSC, we even have $\rho \le N/\ell^2$, see [3, Proposition 5.2] for SC and it can be generalized to the GSC; notice their ρ, k and $k^2 - R$ (in $\mathbb{R}^2$) correspond to our ρ^{-1}, ℓ and $\ell^d - R = N$ respectively.

Finally, we use the standard Sierpinski carpet K in Section 6 to give an illustration of Theorem 1.8. Recall that the boundary S of $\mathcal{M}$ is

$$\{(a,b) : 2a + b = 1, b \ge a\} \cup \{(a,b) : a + 2b = 1, a \ge b\}.$$

Let $\rho \in (0,1)$ be the renormalization factor of the associated Dirichlet form. Let $\mu = (\mu_1, \dots, \mu_8)$ be a self-similar measure on K with $\mu_{2i+1} = \mu_1$, $\mu_{2i+2} = \mu_2$ for $i = 1,2,3$. Let β be the unique positive number satisfying

$$(\max\{\rho\mu_1, \rho\mu_2\})^{\frac{1}{\beta}} + 2\,(\min\{\rho\mu_1, \rho\mu_2\})^{\frac{1}{\beta}} = 1.$$

Let $D_{g_{a,b}}$ be the metric associated with weights $a = (\rho\mu_1)^{1/\beta}$ and $b = (\rho\mu_2)^{1/\beta}$. Then $(a,b) \in S$, and the time change Brownian motion on K via μ has a continuous heat kernel $p_t(x,y)$ satisfying the estimate as in (1.1) with the metric $d(\cdot,\cdot)$ given by $D_{g_{a,b}}(\cdot,\cdot)$.

8. Completing the Proof of Lemma 4.3

We now prove the sublemma of Lemma 4.3.

Sublemma. *For* $\boldsymbol{a} \in (0,1)^k$, $\sup_{n\ge 0} D_g^{(n)}(L,R) < \infty \;\Leftrightarrow\; \sup_{n\ge 0} D_g^{(n)}(p_1,p_2) < \infty.$

Without loss of generality, we will consider the GSC is in $\mathbb{R}^2$; since we are mainly considering the n-chains here, we will omit the superscript n on $\gamma^{(n)}$ when there is no confusion. The direction "$\Leftarrow$" is trivial, let us prove the direction "$\Rightarrow$".

Assume $\sup_{n\ge 0} D_g^{(n)}(L,R) < \infty$, and denote the value by M. Hence for each $n \ge 0$, there is an n-chain γ starting from L and ending at R such that $g(\gamma) \le M$. We will show that there is another n-chain γ' joining p_1 and p_2 such that $g(\gamma') \le CM$ and C is independent of n. Then $\sup_{n\ge 0} D_g^{(n)}(p_1,p_2) < \infty$. We divide our proof into two steps in the sequel.

First we specify three types of transformations we will need in the construction. Let S denote the unit square, and consider $F_w(S)$; we use the same L and R to denote the left and right side of $F_w(S)$. Let ϑ be an n-chain in $F_w(S)$ cross over L to R (or the other two sides), we call ϑ' a *G-image* of ϑ if it is one of the following:

(i) (*Symmetric image*) $\vartheta' = \sigma(\vartheta)$ for $\sigma \in G$. In particular, we use τ to denote reflecting the chain through to the vertical bisector of $F_w(S)$.

(ii) (*Deflected image*) Let ϑ get across a diagonal separating L and R of $F_w(S)$, then we can keep one portion of the chain before the cross, and deflect the other portion along the diagonal to get a new chain ϑ' that reaches the upper or lower sides of $F_w(S)$.

(iii) (*reflected image to neighbor*) Let $F_{w'}(S)$ be the neighbor of K_w that touches θ, we can reflect the chain in $F_w(S)$ to $F_{w'}(S)$ along the intersecting edge.

Note that in (i), (ii), ϑ' and ϑ in K_w satisfies

$$g(\vartheta') = g(\vartheta); \tag{8.1}$$

and in (iii), we have (by Lemma 2.3),

$$g(\theta) = \frac{a}{a'} g(\theta'). \tag{8.2}$$

Remark 1. In the proof, we also allow the G-transform to act on $F_w(S)$ to some other $F'_w(S)$ by repeatedly using (i)-(iii) and a scaling. In this case, the n-chain is transformed to be an $n + (|w'| - |w|)$-chain.

Remark 2. We need one more technique in the construction: Let θ be an n-chain in $F_w(K)$ with left and right end cells x and y respectively; let θ' be another n-chain with right end cell y'. Consider the intersection of the two chains (use (ii) starting form y' if necessary), we can produce a new n-chain θ'' by taking θ before the intersection and θ' after the intersection. Then θ'' starts from x and ends at y', and $g(\theta'') \le g(\theta) + g(\theta')$.

Let γ be an n-chain in K. Let γ_L and γ_R denote the left half and right half of the chain, divided by the vertical bisector of S. Then

$$g(\gamma_L) \le \frac{1}{2} g(\gamma) \quad \text{or} \quad g(\gamma_R) \le \frac{1}{2} g(\gamma).$$

We denote the two cases by Case (A) and Case (B) respectively. By similarity, this also applied to $F_w(\mathcal{S})$.

Let $\gamma = (w(1), w(2), \cdots)$ be the fixed n-chain as above. Let $\Sigma_1 := \{i \in \Sigma : F_i(K) \cap L \neq \emptyset\}$ be the indices of 1-cells at the outer border of K. Denote by $a_{\min} = \min\{a_i : i \in \Sigma_1\}$, and let i_0 be the index that attains the minimum. We denote by p_{i_0} the fixed point of F_{i_0}, i.e., $F_{i_0}(x) = \rho(x - p_{i_0}) + p_{i_0}$.

Step 1.

We will construct an n-chain γ' between p_{i_0} and $\tau(p_{i_0})$ such that $g(\gamma') \le C' g(\gamma)$ for some $C' > 0$ independent of n. This gives $\sup_{n \ge 0} D_g^{(n)}(p_{i_0}, \tau(p_{i_0})) < \infty$.

By applying τ on S, we can assume that γ satisfies Case (A). Let K_{u_1} be the 1-cell contain $K_{w(1)}$; let S_1 be the rectangle that is the union of the 1-cells that intersects L. Let j_0 be the first $w(j)$ that exits S_1, and let γ_1 be the segment before j_0 that is in S_1. Using the G-transform (iii), we can "fold up" the γ_1 into to K_{u_1}, and denote it by $\widetilde{\gamma}_1$; obviously $\widetilde{\gamma}_1$ is a n-chain in K_{u_1} from $F_{u_1}(L)$ to $F_{u_1}(R)$ and satisfies

$$g(\widetilde{\gamma}_1) \le \frac{a_{u_1}}{a_{\min}} g(\gamma_1) \le \frac{a_{u_1}}{2a_{\min}} g(\gamma). \tag{8.3}$$

On K_{u_1}, similar to the situation of γ in K, there are also Case(A) and Case (B) for $\widetilde{\gamma}_1$.

Now let

$$K_{i_0} \sim K_{t_1} \sim \cdots \sim K_{t_{\kappa_1}} \sim K_{\sigma(i_0)}, \quad t_j \in \Sigma \tag{8.4}$$

be a finite sequence of distinct 1-cells connecting K_{i_0} and $K_{\sigma(i_0)}$. Consider the G-images of $\widetilde{\gamma}_1$ to K_{i_0}. In Case (A), we use this G-image in each of the 1-cell of $\{K_{t_j}\}_{j=1}^{\kappa_1}$ (not include the first and the last cell), and apply the suitable G-transform (i) or (ii) to paste up these segments to form a connected n-chain ζ_1 in $\{K_{t_j}\}_{j=1}^{\kappa_1}$; for Case (B), we use τ on K_{t_1} (i.e., reflecting the G-image on the vertical bisector of K_{t_1}), and use the same construction to get ζ_1. In either cases we have the estimate (using (iii))

$$g(\zeta_1) \le \kappa_1 \frac{g(i_0)}{g(u_1)} g(\tilde{\gamma}_1) = \kappa_1 \frac{a_{\min}}{a_{u_1}} g(\tilde{\gamma}_1).$$

To fix our mind, let us assume our ζ_1 comes from Case (A). We reflect the first 2-cell of ζ_1 to the left and call it $K_{i_0 u_2}$, and the n-chain there as $\tilde{\gamma}_2$. Note that by our choice of the two cases, we have

$$g(\tilde{\gamma}_2) \ \le \ \frac{1}{2^2} \frac{a_1 a_2}{a_{\min}^2} g(\gamma).$$

We use it to construct a 2-chain ζ_2 to reach $K_{i_0^2}$ (running in the opposite direction). Consider a chain of distinct 2-cells

$$K_{i_0 u_2} = K_{i_0 s_1} \sim K_{i_0 s_2} \sim \cdots \sim K_{i_0 s_{\kappa_2}} \sim K_{i_0 i_0}, \quad s_j \in \Sigma.$$

We apply the same construction to obtain an n-chain in $\{K_{u_0 s_j}\}_{j=1}^{\kappa_2}$, and an estimate

$$g(\zeta_2) \le \kappa_2 \frac{g(i_0^2)}{g(u_1 u_2)} g(\tilde{\gamma}_1) = \kappa_2 \frac{a_{\min}^2}{a_{u_1 u_2}} g(\tilde{\gamma}_1).$$

Next we continue extending ζ_2 to ζ_3 in $K_{i_0^3}$, we will face the same Case (A), Case (B) situation in $K_{i_0^2}$. We proceed as the above to choose $\tilde{\gamma}_3$ and construct ζ_3. But we need to be caution in Case (B), the reflecting case. Nevertheless, an application of Remark 2 on $K_{i_0^2}$ will alow us to connect ζ_2 and ζ_3.

We apply the same construction through $3 \le \ell \le n$, we have

$$g(\widetilde{\gamma}_\ell) \le \frac{1}{2^\ell} \cdot \frac{a_{u_1 \cdots u_\ell}}{a_{\min}^\ell} g(\gamma).$$

Note that the κ_ℓ, $1 \le \ell \le n$ is uniformly bounded, denote the bound by C. Then

$$g(\zeta_\ell) \le C \frac{g(i_0^\ell)}{g(u_1 \cdots u_\ell)} \cdot g(\widetilde{\gamma}_\ell) = C \cdot \frac{a_{\min}^\ell}{a_{u_1 \cdots u_\ell}} \cdot g(\widetilde{\gamma}_\ell).$$

It follows that

Finally, we concatenate $\{\zeta_n, \cdots, \zeta_1\}$ to form an n-chain in $K_{i_0} \cup \{K_{t_j}\}_{j=1}^{\kappa_1}$. By reflection the part before the vertical bisector of S to the right side, and put these two parts together, we get a new n-chain starting at p_{i_0} and ending at $p_{\sigma(i_0)}$. We denote it by γ', it satisfies

$$g(\gamma') \leq C' \sum_{\ell=1}^{n} g(\zeta_\ell) \leq C' \sum_{\ell=1}^{n} \frac{1}{2^\ell} g(\gamma) \leq C'' g(\gamma)$$

for some C' independent of n. Hence $\sup_{n\geq 0} D_g^{(n)}(p_{i_0}, \sigma(p_{i_0})) < \infty$.

Step 2. Let $M = \sup_{n\geq} g(\gamma'^{(n)})$, where $\gamma'^{(n)}$ is the n-chain constructed in Step 1 (the superscript n was suppressed there for simplicity, but we will keep it here). For each n, we will use $\left\{\gamma'^{(n-i)}\right\}_{i=2}^{n-1}$ in the following construction; note that they are all start from p_0 and ends at $\tau(p_0)$.

Define $q_i = F_{1^i}(\tau(p_{i_0}))$ where F_1 is the similitude with fixed point p_1. We will use the $\left\{\gamma'^{(n-i)}\right\}_{n\geq 0}$ to construct an n-chain $\xi^{(n)}$ connecting q_i's consecutively for $i = 1, 2, \ldots, n-2$ (and then p_1) such that $g(\xi^{(n)}) \leq CM$ for some $C > 0$ independent of n. This yields

$$\sup_{n\geq 0} g(\xi^{(n)}) \leq CM.$$

First, let q_1 and q_2 be connected by some 2-cells in K_1 between K_{1^2} and $K_{1\tau(i_0)}$ (as in (8.4)). Considering $F_{1^2}(\gamma'^{(n-2)})$ as an n-chain in K_{1^2}, we can use G-images of it to construct an n-chain $\xi_1^{(n)}$ in those the 2-cells between q_1 and q_2 (the same construction of $\zeta_1^{(n)}$ as in Step 1), and use Remark 2 to ensure ξ_1 reaches q_1. Then as before, there is $C' > 0$ (only dependent on K and $\boldsymbol{a}$) such that

$$g(\xi_1^{(n)}) \leq C' \cdot g(1) \cdot g(\gamma'^{(n-2)}) \leq C_1 a_1 M.$$

Inductively, for $1 \leq i \leq n-2$, by using $\gamma'^{(n-2-i)}$, we get an n-chain $\xi_i^{(n)}$ between q_i and q_{i+1} inside K_{1^i}, and

$$g(\xi_i^{(n)}) \leq C' \cdot g(1^i) \cdot g(\gamma'^{(n-1-i)}) \leq C_1 a_1^i M.$$

Finally, q_1 and q_{n-2} can be connected by some n-cells with bounded total weight; we can trivially add several n-cells to connect q_{n-2} to p_1 . Hence p_1 can be connected to q_1 and satisfies

$$g(\xi^n) \leq C_1 M \sum_{i=1}^{n-2} a_1^i + C'.$$

Denote this chain by ξ. Let $q_1' = \tau(q_1)$, we can reflect ξ to obtain an n-chain to connect q_1' and p_2. Also we can show that q_1 connects q_1' by using the G-transform of $\gamma'^{(n-1)}$ (with a different constant). Combining the three parts, we obtain an n-chain $\xi^{(n)}$ joining p_1 and p_2 with bound $\leq CM$. This implies

$$\sup_{n\geq 0} D_g^{(n)}(p_1, p_2) < \infty,$$

which proves the sublemma. □

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Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, China
Email address: 001gqs@163.com

Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, China, & Department of Mathematics, University of Pittsburgh, Pittsburgh, Pa. 15217, U.S.A.
Email address: kslau@math.cuhk.edu.hk

Department of Mathematics, Nanjing University, Nanjing 210093, China
Email address: huaqiu@nju.edu.cn

School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China
Email address: ruanhj@zju.edu.cn