

SOBOLEV SPACES ON P.C.F. SELF-SIMILAR SETS I: CRITICAL ORDERS AND ATOMIC DECOMPOSITIONS

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ABSTRACT. We consider the Sobolev type spaces $H^\sigma(K)$ with $\sigma \geq 0$, where K is a post-critically finite self-similar set with the natural boundary. Firstly, we compare different classes of Sobolev spaces $H_N^\sigma(K)$, $H_D^\sigma(K)$ and $H^\sigma(K)$, and observe a sequence of critical orders of σ in our comparison theorem. Secondly, We present a general atomic decomposition theorem of Sobolev spaces $H^\sigma(K)$, where the same critical orders play an important role. At the same time, we provide purely analytic approaches for various Besov type characterizations of Sobolev spaces $H^\sigma(K)$.

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1. INTRODUCTION

Analysis on fractals, based on the construction of Laplacians and Dirichlet forms, has been studied for years. In this paper, we study function spaces on post-critically finite (p.c.f.) self-similar sets, based on Kigami's construction of Dirichlet forms ([22, 23]). See backgrounds in books [24, 37], and see [4] for the probabilistic approach. One of the most well-known example of p.c.f. self-similar sets is the Sierpinski gasket (\mathcal{SG}), see Figure 1.

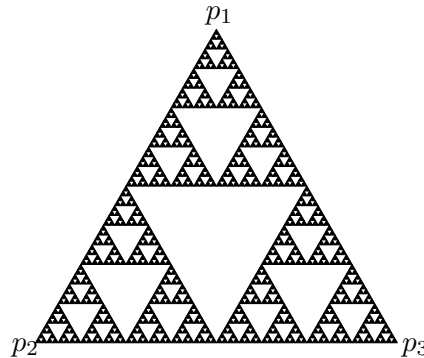


FIGURE 1. the Sierpinski gasket.

In 2003, on a p.c.f. fractal K , Strichartz[35] considered general Sobolev type spaces $L_\sigma^p(K)$ and Besov type spaces $\Lambda_\sigma^{p,q}(K)$ with $1 \leq p, q \leq \infty$ and $\sigma \geq 0$, and gave an all-round study on various embedding theorems and interpolation results. A systematical introduction of Sobolev spaces can also be found in [11] on more general metric measure spaces by Grigor'yan. We will focus on the important case $p = q = 2$ in this paper, and use the notation $H^\sigma(K)$ for these L^2 Sobolev spaces. In the definition, the p.c.f. self-similar K is viewed as a domain with the natural boundary, which consists of finitely many points. For example, the Sierpinski gasket \mathcal{SG} has the boundary $V_0 = \{p_1, p_2, p_3\}$, the three vertices of a triangle, as shown in Figure 1.

On the other hand, one may define the Sobolev spaces $H_N^\sigma(K)$ with the Bessel potential $(1 - \Delta_N)^{-\sigma/2}$, where Δ_N is the Neumann Laplacian, defined as the generator of the heat semigroup, see [18, 35] for example, without involving the boundary from definition. Similarly, one can define another class of Sobolev spaces $H_D^\sigma(K)$ using the Dirichlet Laplacian Δ_D or the related Brownian motion killed at the boundary.

It is of interest to compare the distinct classes of Sobolev spaces. As one of the main results in this paper, we obtain a full comparison of these spaces, see Theorem 3.2. Below is the part concerning the relationship between $H^\sigma(K)$ and $H_N^\sigma(K)$.

Theorem 1. For $k \in \mathbb{Z}_+$, define $\mathcal{H}'_{k-1} = \{f : \Delta^k f = \text{constant}, \int_K f d\mu = 0\}$, and d_S be the spectral dimension of Δ_N . Then, for $\sigma \in (2k - \frac{d_S}{2}, 2k + 2 - \frac{d_S}{2}) \cap \mathbb{R}_+$,

$$H^\sigma(K) = H_N^\sigma(K) \oplus \mathcal{H}'_{k-1};$$

for $\sigma = 2k + 2 - \frac{d_S}{2}$, $H_N^\sigma(K)$ is not closed in $H^\sigma(K)$ and

$$H^\sigma(K) = \text{cl}(H_N^\sigma(K)) \oplus \mathcal{H}'_{k-1}.$$

From Theorem 3.2, one can easily observe the critical orders $2k + \frac{d_S}{2}$ and $2k + 2 - \frac{d_S}{2}$ with $k \in \mathbb{Z}_+$, where d_S is the spectral dimension. See [25] for a discussion on the spectral dimension of Δ_N or Δ_D . In fact, we have $\frac{d_S}{2} = \frac{d_H}{1+d_H}$, where d_H is the Hausdorff dimension of K with respect to the effective resistance metric (see [24, 37]), and $d_W := 1 + d_H$ is the walk dimension (see [14, 26]). In particular, on the Sierpinski gasket \mathcal{SG} equipped with the standard energy and measure, we have $d_H = \frac{\log 3}{\log 5/3}$, $d_W = \frac{\log 5}{\log 5/3}$, $d_S = \frac{2 \log 3}{\log 5}$, and the critical orders are $\frac{\log 3}{\log 5} + 2\mathbb{Z}_+$ and $2 - \frac{\log 3}{\log 5} + 2\mathbb{Z}_+$.

The next topic of this paper is the characterization of the Sobolev spaces, and we are interested in the role of the critical orders. In fact, as a particular case, we usually write a function in $\text{dom}\mathcal{E}$ (which equals $H_N^1(K)$, and also $H^1(K)$ by Theorem 3.2) as a series of tent functions, which is helpful in various cases, for example the trace theorem of Jonsson [21] for \mathcal{SG} . We will extend this characterization to general Sobolev spaces. See Theorem 4.1.

Here we use the Sierpinski gasket \mathcal{SG} as an illustration. We have for $0 \leq \sigma < \frac{\log 3}{\log 5}$, each function f in $H^\sigma(\mathcal{SG})$ admits a unique expansion

$$f = C + \sum_{w \in W_*} \sum_{i=1}^3 a_{wi} \chi_{F_{wi} \mathcal{SG}}, \quad \text{with } C \in \mathbb{R}, a_{wi} \in \mathbb{R}, \sum_{i=1}^3 a_{wi} = 0,$$

and $\|f\|_{H^\sigma(\mathcal{SG})} \asymp (|C|^2 + \sum_{m=0}^{\infty} 3^{-m} 5^{\sigma m} \sum_{w \in W_m} \sum_{i=1}^3 |a_{wi}|^2)^{1/2}$;

for $\frac{\log 3}{\log 5} < \sigma < 2 - \frac{\log 3}{\log 5}$, f admits

$$f = h + \sum_{x \in V_* \setminus V_0} c_x \psi_x, \quad \text{with } h \in \mathcal{H}_0, c_x \in \mathbb{R},$$

and $\|f\|_{H^\sigma(\mathcal{SG})} \asymp (\|h\|_{L^2(\mathcal{SG})}^2 + \sum_{m=0}^{\infty} 3^{-m} 5^{\sigma m} \sum_{x \in V_{m+1} \setminus V_m} |c_x|^2)^{1/2}$; while the expansions can be extended to the case $\sigma \geq 2$ by repeatedly applying the Green's operator. As an application of this result, we invite readers to refer to [9] for the trace spaces of $H^\sigma(\mathcal{SG})$ onto the bottom line segment of \mathcal{SG} , which extends Jonsson's work on $\text{dom}\mathcal{E}$ [21]. Readers may also compare our decomposition theorem with the theorem of multi-harmonic splines on p.c.f. fractals [39].

Lastly, we consider the equivalence of Sobolev spaces $H^\sigma(K)$ with some Besov type spaces on K .

In 1996, still for the Sierpinski gasket \mathcal{SG} , Jonsson [20] obtained that the standard energy domain $\text{dom}\mathcal{E}$ can be characterized to be a Besov type space $B_1^{2,\infty}(\mathcal{SG})$. Later, the result was generalized to nested fractals by Pietruska-Pařuba [30] in terms of Euclidean metric and normalized Hausdorff measure, and to p.c.f. fractals by Hu and Wang [17] by using the effective resistance metric and an associated d -regular measure instead.

In 2005, Hu and Zähle[18] studied $H_N^\sigma(K)$, which generalizes $\text{dom}\mathcal{E}$, on general metric measure spaces, with the assumption of two-sided heat kernel estimates, which was confirmed to be true for p.c.f. fractals in terms of the effective resistance metric by Hambly and Kumagai, Kumagai and Sturm (see [14, 26] where probabilistic techniques are much involved). The combination of results in [18, 14, 26] yields that, for a p.c.f. fractal K with a regular harmonic structure, $H_N^\sigma(K)$ is equivalent to a Besov type space $B_\sigma^{2,2}(K)$ for $0 < \sigma < 1$, as well as $H_N^1(K) = \text{dom}\mathcal{E} = B_1^{2,\infty}(K)$. We refer to [7, 8, 12] for Besov type characterizations of Sobolev spaces with $0 < \sigma < 1$ on general metric measure spaces, related to the heat kernel estimates, and [8, 10] for related interpolation results.

It is of interest to find a direct analytical way to characterize a similar Besov type characterization of $H^\sigma(K)$ on p.c.f. fractals. As a particular situation, [20, 30, 17] provided the analytical approach for $\text{dom}\mathcal{E} = B_1^{2,\infty}(K)$.

In this paper, we will provide a purely analytic method to show that $H^\sigma(K) = H_N^\sigma(K) = B_\sigma^{2,2}(K)$ for $0 < \sigma < 1$, without using heat kernel estimate assumption, see Theorem 4.9. Also, we will take care of the Besov spaces $\Lambda_\sigma^{2,2}(K)$, and the extension $\tilde{\Lambda}_\sigma^{2,2}(K)$ introduced by Strichartz [35], and prove that they are all equivalent to $H^\sigma(K)$ for suitable σ . See Theorem 4.7, where we show

$$H^\sigma(K) = \Lambda_\sigma^{2,2}(K) \text{ for } \frac{d_S}{2} < \sigma < 1, \text{ and } H^\sigma(K) = \tilde{\Lambda}_\sigma^{2,2}(K) \text{ for } \frac{d_S}{2} < \sigma < 2.$$

We also introduce a new class of Besov type spaces $\Gamma^\sigma(K)$ that equivalent to $H^\sigma(K)$ based on the cell graphs approximating K , see Theorem 6.1. In particular, the case $1 < \sigma < 2$ are not dealt with in [18].

At the end of the introduction, we briefly show the structure of this paper.

In Section 2, for a p.c.f. fractal K , we collect some notations and definitions, including the effective resistance metric R , the d_H -regular measure μ , and the Sobolev spaces $H_D^\sigma(K)$, $H_N^\sigma(K)$ and $H^\sigma(K)$.

In Section 3, we focus on the comparison theorem of various Sobolev spaces on K . We will provide a detailed proof of Theorem 3.2.

In Section 4, we introduce the main theorems concerning the characterizations of Sobolev spaces, including the atomic decompositions (Theorem 4.1), and Besov type characterizations (Theorem 4.7 and 4.9), but postpone the proofs to later sections.

From Section 5 to Section 7, we focus on the characterization of Sobolev spaces with orders $0 \leq \sigma < 1$. In Section 5, we introduce the Besov type spaces $\Gamma^\sigma(K)$ based on the cell graph energies, and discuss the decomposition in terms of Haar functions. In Section 6, we introduce the notion of so-called smoothed Haar functions, and use it as a key tool to prove the characterizations of Sobolev spaces (Theorem 6.1, 6.7 and 4.9). Lastly, in Section 7, we finish the proof of atomic decomposition theorem (Theorem 4.1) for $0 \leq \sigma < 1$. At the same time we prove Theorem 4.7(a) and Theorem 5.9.

Section 8 is parallel to Section 5 to 7, dealing with the characterization of $H^\sigma(K)$ with higher orders $1 \leq \sigma < 2$. Since the idea is very similar, we will only provide the key lemmas in this section, and sketch the proof.

Throughout the paper, we always use the notation $f \lesssim g$ if there is a constant $C > 0$ such that $f \leq Cg$, and write $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.

2. THE DIRICHLET FORMS AND SOBOLEV SPACES ON P.C.F. SELF-SIMILAR SETS

The analysis on p.c.f. self-similar fractals was originally developed by Kigami in [22, 23]. For convenience of readers, in this section, we will first briefly recall the constructions of Dirichlet forms and Laplacians on p.c.f. fractals, then introduce the definition of associated Sobolev spaces. Interested readers please refer to [16, 38] for p -energy and corresponding L^p Sobolev spaces on fractals as an extension.

Let $\{F_i\}_{i=1}^N$ be a finite collection of contractions on a complete metric space (X, d) . The self-similar set associated with the *iterated function system (i.f.s.)* $\{F_i\}_{i=1}^N$ is the unique compact set $K \subset X$ satisfying

$$K = \bigcup_{i=1}^N F_i K.$$

Each copy $F_i K$ is called a *1-cell* of K . For $m \geq 1$, we define $W_m = \{1, \dots, N\}^m$ the collection of *words* of length m , and for each $w \in W_m$, denote

$$F_w = F_{w_1} \circ F_{w_2} \circ \dots \circ F_{w_m}.$$

The set $F_w K$ is called a *m-cell* of K . Set $W_0 = \emptyset$, and let $W_* = \bigcup_{m \geq 0} W_m$ be the collection of all finite words. For $w = w_1 w_2 \dots w_m \in W_* \setminus W_0$, we write $w^* = w_1 w_2 \dots w_{m-1}$ by deleting the last letter of w .

Define the shift space $\Sigma = \{1, 2, \dots, N\}^{\mathbb{N}}$. There is a continuous surjection $\pi : \Sigma \rightarrow K$ defined by

$$\pi(\omega) = \bigcap_{m \geq 1} F_{[\omega]_m} K,$$

where for $\omega = \omega_1 \omega_2 \dots$ in Σ we write $[\omega]_m = \omega_1 \omega_2 \dots \omega_m \in W_m$ for each $m \geq 1$. Let

$$\mathcal{C}_K = \bigcup_{i \neq j} F_i K \cap F_j K, \quad \mathcal{C} = \pi^{-1}(\mathcal{C}_K), \quad \mathcal{P} = \bigcup_{n \geq 1} \sigma^n \mathcal{C},$$

where σ is the shift map define as $\sigma(\omega_1 \omega_2 \dots) = \omega_2 \omega_3 \dots$. \mathcal{P} is called the *post-critical set*. Call K a *p.c.f. self-similar fractal* if $\#\mathcal{P} < \infty$. In what follows, we always assume that K is a connected p.c.f. fractal.

Let $V_0 = \pi(\mathcal{P})$ and call it the *boundary* of K . For $m \geq 1$, we always have $F_w K \cap F_{w'} K \subset F_w V_0 \cap F_{w'} V_0$ for any $w \neq w' \in W_m$. Denote $V_m = \bigcup_{w \in W_m} F_w V_0$ and let $l(V_m) = \{f : f \text{ maps } V_m \text{ into } \mathbb{R}\}$. Write $V_* = \bigcup_{m \geq 0} V_m$.

Let $H = (H_{pq})_{p, q \in V_0}$ be a symmetric linear operator(matrix). H is called a (*discrete*) *Laplacian* on V_0 if H is non-positive definite; $Hu = 0$ if and only if u is constant on V_0 ; and $H_{pq} \geq 0$ for any $p \neq q \in V_0$. Given a Laplacian H on V_0 and a vector $\mathbf{r} = \{r_i\}_{i=1}^N$ with $r_i > 0$, $1 \leq i \leq N$, define the (*discrete*) *Dirichlet form* on V_0 by

$$\mathcal{E}_0(f, g) = -(f, Hg),$$

and inductively on V_m by

$$\mathcal{E}_m(f, g) = \sum_{i=1}^N r_i^{-1} \mathcal{E}_{m-1}(f \circ F_i, g \circ F_i), \quad m \geq 1,$$

for $f, g \in l(V_m)$. Write $\mathcal{E}_m(f, f) = \mathcal{E}_m(f)$ for short.

Say (H, \mathbf{r}) is a *harmonic structure* if for any $f \in l(V_0)$,

$$\mathcal{E}_0(f) = \min\{\mathcal{E}_1(g) : g \in l(V_1), g|_{V_0} = f\}.$$

In this paper, we will always assume that there exists a harmonic structure associated with K , and in addition, $0 < r_i < 1$ for all $1 \leq i \leq N$. Call (H, \mathbf{r}) a *regular harmonic structure* on K .

Now for each $f \in C(K)$, the sequence $\{\mathcal{E}_m(f)\}_{m \geq 0}$ is nondecreasing. Let

$$\mathcal{E}(f, g) = \lim_{m \rightarrow \infty} \mathcal{E}_m(f, g) \text{ and } \text{dom}\mathcal{E} = \{f \in C(K) : \mathcal{E}(f) < \infty\}, \quad (2.1)$$

where $f, g \in C(K)$ and we write $\mathcal{E}(f) := \mathcal{E}(f, f)$ for short. Call $\mathcal{E}(f)$ the *energy* of f . It is known that $(\mathcal{E}, \text{dom}\mathcal{E})$ turns out to be a local regular Dirichlet form on $L^2(K, \mu)$ for any Radon measure μ on K . See Section 2.4 and Section 3.4 of the book [24] for details.

An important feature of the form $(\mathcal{E}, \text{dom}\mathcal{E})$ is the *self-similar identity*

$$\mathcal{E}(f, g) = \sum_{i=1}^N r_i^{-1} \mathcal{E}(f \circ F_i, g \circ F_i), \quad \forall f, g \in \text{dom}\mathcal{E}. \quad (2.2)$$

Furthermore, denote $r_w = r_{w_1} r_{w_2} \cdots r_{w_m}$ for each $w \in W_m, m \geq 0$. Then for $m \geq 1$, we have

$$\mathcal{E}_m(f, g) = \sum_{w \in W_m} r_w^{-1} \mathcal{E}_0(f \circ F_w, g \circ F_w), \quad \mathcal{E}(f, g) = \sum_{w \in W_m} r_w^{-1} \mathcal{E}(f \circ F_w, g \circ F_w).$$

See [24] and [37] for details and any unexplained notations.

2.1. Resistance metric and self-similar measure. To study the Besov spaces on K , we need a suitable metric and a comparable measure. Instead of the original d , a natural choice of metric is the effective resistance metric $R(\cdot, \cdot)$ [24], which matches the form $(\mathcal{E}, \text{dom}\mathcal{E})$.

Definition 2.1. For $x, y \in K$, the *effective resistance metric* $R(x, y)$ between x and y is defined by

$$R(x, y)^{-1} = \min\{\mathcal{E}(f) : f \in \text{dom}\mathcal{E}, f(x) = 0, f(y) = 1\}.$$

It is known that R is indeed a metric on K which is topologically equivalent to the metric d , and for $w \in W_*$, we always have $\text{diam}(F_w K) \asymp r_w$, where $\text{diam}(F_w K) = \max\{R(x, y) : x, y \in F_w K\}$.

We will always choose the following self-similar measure μ on K .

Definition 2.2. Let μ be the unique self-similar measure on K satisfying

$$\mu = \sum_{i=1}^N r_i^{d_H} \mu \circ F_i^{-1},$$

and $\mu(K) = 1$, where d_H is determined by the equation $\sum_{i=1}^N r_i^{d_H} = 1$.

For $x \in K$, $\rho > 0$, denote $B(x, \rho) = \{y \in K : R(x, y) < \rho\}$ the *ball* centered at x with radius ρ . From now on, for simplicity, we will always write $L^2(K)$ instead of $L^2(K, d\mu)$, and do similarly for the Sobolev spaces to be defined.

It is well known that the measure μ is comparable with R . More precisely, the following estimate is proven in [4], Proposition 8.9.

Proposition 2.3. For any $x \in K$, $0 < \rho \leq 1$, we have $\mu(B(x, \rho)) \asymp \rho^{d_H}$.

In this paper, we will consider cells of comparable sizes. We introduce the following notations for convenience.

Definition 2.4. For $0 < t \leq 1$, define

$$\Lambda(t) = \{w \in W_* : r_w \leq t < r_{w^*}\}.$$

In particular, set $r = \min_{i=1}^N r_i$, and let $\Lambda_m = \Lambda(r^m)$ for $m \geq 0$.

Obviously, $\{\Lambda_m\}_{m \geq 0}$ provides a nested partition of K satisfying that

$$K = \bigcup_{w \in \Lambda_m} F_w K, \quad F_w K \cap F_{w'} K \subset F_w V_0 \cap F_{w'} V_0, \quad \forall w, w' \in \Lambda_m,$$

and thus the self-similar identity (2.2) extends to

$$\mathcal{E}(f, g) = \sum_{w \in \Lambda_m} r_w^{-1} \mathcal{E}(f \circ F_w, g \circ F_w), \quad \forall f, g \in \text{dom} \mathcal{E}.$$

2.2. Sobolev spaces $H^\sigma(K)$, $H_D^\sigma(K)$ and $H_N^\sigma(K)$. We start from Sobolev spaces $H^\sigma(K)$ with integer orders, then extend to fractional orders using complex interpolation. Readers may refer to [19, 21, 32, 33, 35] and the references therein for related works, such as bump functions, trace theorems, pseudo-differential operators and a distribution theory on p.c.f. fractals.

We start with the definition of Laplacians.

Definition 2.5. Let $\text{dom}_0 \mathcal{E} = \{\varphi \in \text{dom} \mathcal{E} : \varphi|_{V_0} = 0\}$.

(a). For $f \in \text{dom} \mathcal{E}$, say $\Delta f = u$ if

$$\mathcal{E}(f, \varphi) = - \int_K u \varphi d\mu, \quad \forall \varphi \in \text{dom}_0 \mathcal{E}$$

(b). For $f \in \text{dom}_0 \mathcal{E}$, say $\Delta_D f = u$ if

$$\mathcal{E}(f, \varphi) = - \int_K u \varphi d\mu, \quad \forall \varphi \in \text{dom}_0 \mathcal{E},$$

(c). For $f \in \text{dom} \mathcal{E}$, say $\Delta_N f = u$ if

$$\mathcal{E}(f, \varphi) = - \int_K u \varphi d\mu, \quad \forall \varphi \in \text{dom} \mathcal{E}.$$

On $L^2(K)$, both Δ_D and Δ_N are non-positive definite self-adjoint operators, and Δ is a closed operator such that $\Delta_D \subset \Delta, \Delta_N \subset \Delta$. In addition, Δ_D is invertible, and $G = -\Delta_D^{-1}$ can be realized with the Green's function $G(x, y) \in C(K \times K)$, i.e.

$$Gf = \int_K G(x, y) f(y) d\mu(y).$$

Clearly, we have $-\Delta Gf = f, \forall f \in L^2(K)$. For further discussions on the Green's operator G , see books [24] and [37].

In the following, we define three different classes of Sobolev spaces, associated with the different Laplacians. One of our main interest in this paper is to clarify their relationships.

Definition 2.6. (a). For $k \in \mathbb{Z}_+$, define the Sobolev space $H^{2k}(K)$ as

$$H^{2k}(K) = \{f \in L^2(K) : \Delta^j f \in L^2(K) \text{ for all } j \leq k\}$$

with the norm of f given by $\|f\|_{H^{2k}(K)} = \sum_{j=0}^k \|\Delta^j f\|_{L^2(K)}$.

For $0 < \theta < 1$, $k \in \mathbb{Z}_+$, define $H^{2k+2\theta}(K)$ by using complex interpolation,

$$H^{2k+2\theta}(K) = [H^{2k}(K), H^{2k+2}(K)]_\theta.$$

(b). For $\sigma \geq 0$, define

$$H_D^\sigma(K) = (I - \Delta_D)^{-\sigma/2} L^2(K),$$

with norm $\|f\|_{H_D^\sigma(K)} = \|(I - \Delta_D)^{\sigma/2} f\|_{L^2(K)}$.

(c). For $\sigma \geq 0$, define

$$H_N^\sigma(K) = (I - \Delta_N)^{-\sigma/2} L^2(K),$$

with norm $\|f\|_{H_N^\sigma(K)} = \|(I - \Delta_N)^{\sigma/2} f\|_{L^2(K)}$.

Remark. $H_D^\sigma(K)$ and $H_N^\sigma(K)$ are derived from Bessel potentials in a manner analogous to the classical results of Aronszajn and Simith [3], while the classical analog of the space $H^\sigma(K)$ defined by complex interpolation is due to Calderón [6] and Lions [28] separately. See the connection between these spaces in the classical setting in the book [29] by Lions and Magenes.

3. THE CRITICAL ORDERS AND A COMPARISON THEOREM

In this section, we focus on comparing the different Sobolev spaces $H^\sigma(K)$, $H_D^\sigma(K)$ and $H_N^\sigma(K)$ with $\sigma \geq 0$ defined in Section 2.

Let's start from the simplest case, when $\sigma = 2k$ for some $k \in \mathbb{Z}_+$.

For $k \in \mathbb{N}$, define the space of k -multiharmonic functions as

$$\mathcal{H}_{k-1} = \{f : \Delta^k f = 0\},$$

and define

$$\mathcal{H}'_{k-1} = \{f : \Delta^k f = \text{constant}, \int_K f d\mu = 0\}.$$

Both \mathcal{H}_{k-1} and \mathcal{H}'_{k-1} are spaces of dimension $k\#V_0$. Set $\mathcal{H}_{-1} = \mathcal{H}'_{-1} = \{0\}$ for uniformity.

Proposition 3.1. *Let $k \in \mathbb{Z}_+$. Then we have*

$$H^{2k}(K) = H_D^{2k}(K) \oplus \mathcal{H}_{k-1}, \quad H^{2k}(K) = H_N^{2k}(K) \oplus \mathcal{H}'_{k-1}.$$

Proof. It is easy to verify that $H_D^{2k}(K) \cap \mathcal{H}_{k-1} = \{0\}$, $H_N^{2k}(K) \cap \mathcal{H}'_{k-1} = \{0\}$, and

$$H_D^{2k}(K) \oplus \mathcal{H}_{k-1} \subset H^{2k}(K), \quad H_N^{2k}(K) \oplus \mathcal{H}'_{k-1} \subset H^{2k}(K).$$

Next, we show $H^{2k}(K) \subset H_D^{2k}(K) \oplus \mathcal{H}_{k-1}$. Let $f \in H^{2k}(K)$. Define $g = (-1)^k G^k f$ and $h = f - g$. Then, $g \in H_D^{2k}(K)$ and $h \in \mathcal{H}_{k-1}(K)$. Thus, $f \in H_D^{2k}(K) \oplus \mathcal{H}_{k-1}$.

Last, we show $H^{2k}(K) \subset H_N^{2k}(K) \oplus \mathcal{H}'_{k-1}$. It is well-known that the nonzero eigenvalues of $-\Delta_N$ are bounded away from 0 and $\ker \Delta_N = \text{constants}$, so we may define

$$G_N u = \sum_{i=1}^{\infty} \lambda_i^{-1} u_i \langle u, u_i \rangle, \quad \forall u \in L^2(K), \quad (3.1)$$

where $\{u_i\}_{i=1}^\infty \cup \{1\}$ are eigenfunctions of $-\Delta_N$ (chosen to form an orthonormal basis of $L^2(K)$) and λ_i 's are the corresponding eigenvalues. Let $g = G_N^k \Delta^k f + \int_K f d\mu$ and $h = f - g$, we can see that $\Delta^k h = \int_K \Delta^k f d\mu$ and $\int_K h d\mu = 0$, so $h \in \mathcal{H}'_{k-1}$. Thus, $f \in H_N^{2k}(K) \oplus \mathcal{H}'_{k-1}$. \square

In this section, we will extend Proposition 3.1 to the following Theorem 3.2. In particular, for the Dirichlet case, we observe a sequence of critical orders that divides \mathbb{R}_+ into a sequence of open intervals, such that for σ in the k -th interval, it holds that $H^\sigma(K) = H_D^\sigma(K) \oplus \mathcal{H}_{k-1}$, and for σ being a critical order, the relationship will be more complicated. The Neumann case is similar, but with a different sequence of critical orders.

Theorem 3.2. *Let $d_S = \frac{2d_H}{1+d_H}$ which is the spectral dimension of K . We have*

(a). *For $k \geq 0$ and $\sigma \in (2k - 2 + \frac{d_S}{2}, 2k + \frac{d_S}{2}) \cap \mathbb{R}_+$,*

$$H^\sigma(K) = H_D^\sigma(K) \oplus \mathcal{H}_{k-1};$$

for $\sigma = 2k + \frac{d_S}{2}$, $H_D^\sigma(K)$ is not closed in $H^\sigma(K)$ and

$$H^\sigma(K) = cl(H_D^\sigma(K)) \oplus \mathcal{H}_{k-1}.$$

(b). *For $k \geq 0$ and $\sigma \in (2k - \frac{d_S}{2}, 2k + 2 - \frac{d_S}{2}) \cap \mathbb{R}_+$,*

$$H^\sigma(K) = H_N^\sigma(K) \oplus \mathcal{H}'_{k-1};$$

for $\sigma = 2k + 2 - \frac{d_S}{2}$, $H_N^\sigma(K)$ is not closed in $H^\sigma(K)$ and

$$H^\sigma(K) = cl(H_N^\sigma(K)) \oplus \mathcal{H}'_{k-1}.$$

As an immediate consequence of Theorem 3.2, we have the following useful corollary.

Corollary 3.3. $H^1(K) = H_D^1(K) \oplus \mathcal{H}_0 = H_N^1(K) = dom \mathcal{E}$.

We will prove Theorem 3.2 in the rest of this section, and break the proof into several lemmas. Since the proof of part (a) and part (b) are very similar, we will focus on the proof of (a), and sketch the proof of (b) at the same time.

3.1. The sequence spaces. As an important tool, we introduce the following sequence spaces. Throughout this section, we always use the symbol $\alpha = (\alpha_1, \alpha_2, \dots)$ to denote a sequence. Recall that $d_W = 1 + d_H$ and $d_S = \frac{2d_H}{d_W}$. For $\sigma \geq 0$, we denote by $\lambda = \lambda(\sigma) = r^{(d_H - \sigma d_W)/2}$ for short, where $r = \min_{i=1}^N r_i$. Let

$$\begin{aligned} S^\sigma &= \{ \alpha : \{ \lambda^m (\alpha_{m+1} - \alpha_m) \}_{m \geq 1} \in l^2 \}, \\ \tilde{S}^\sigma &= \{ \alpha : \{ \lambda^m \alpha_m \}_{m \geq 1} \in l^2 \}, \end{aligned}$$

with norms

$$\begin{aligned} \|\alpha\|_{S^\sigma} &= |\alpha_1| + \|\lambda^m (\alpha_{m+1} - \alpha_m)\|_{l^2}, \\ \|\alpha\|_{\tilde{S}^\sigma} &= \|\lambda^m \alpha_m\|_{l^2}, \end{aligned}$$

respectively.

The spaces \tilde{S}^σ , $\sigma \in \mathbb{R}$ are stable under complex interpolation, i.e.

$$[\tilde{S}^{\sigma_1}, \tilde{S}^{\sigma_2}]_\theta = \tilde{S}^{(1-\theta)\sigma_1 + \theta\sigma_2}, \quad \forall \sigma_1, \sigma_2 \in \mathbb{R}, \forall \theta \in (0, 1). \quad (3.2)$$

Readers can read [5] Section 5.6 for a proof for more general sequence spaces).

The spaces S^σ looks more complicated, but there is an obvious isomorphism $S^\sigma \rightarrow \tilde{S}^\sigma : \{\alpha_m\}_{m \geq 1} \rightarrow \{\alpha_{m+1} - \alpha_m\}_{m \geq 1}$. Thus, we also have

$$[S^{\sigma_1}, S^{\sigma_2}]_\theta = S^{(1-\theta)\sigma_1 + \theta\sigma_2}, \quad \forall \sigma_1, \sigma_2 \in \mathbb{R}, \forall \theta \in (0, 1). \quad (3.3)$$

Our proof of Theorem 3.2 heavily relies on the above facts about complex interpolation, and the following comparison lemmas of S^σ and \tilde{S}^σ .

Lemma 3.4. (a). For $\sigma < \frac{d_S}{2}$, we have $S^\sigma = \tilde{S}^\sigma$ with $\|\alpha\|_{S^\sigma} \asymp \|\alpha\|_{\tilde{S}^\sigma}$.

(b) For $\sigma > \frac{d_S}{2}$, we have $S^\sigma = \text{constants} \oplus \tilde{S}^\sigma$ with $\|\alpha\|_{S^\sigma} \asymp |c| + \|\tilde{\alpha}\|_{\tilde{S}^\sigma}$, where $c = \lim_{m \rightarrow \infty} \alpha_m$ and $\tilde{\alpha}_m = \alpha_m - c$.

(c). For $\sigma = \frac{d_S}{2}$, \tilde{S}^σ is not a closed subspace of S^σ , and \tilde{S}^σ is dense in S^σ .

Proof. (a). Let $\sigma < \frac{d_S}{2}$ and so $\lambda < 1$. For any $\alpha \in S^\sigma$, by Minkowski inequality, we have

$$\begin{aligned} \|\alpha\|_{\tilde{S}^\sigma} &= \|\lambda^m \alpha_m\|_{l^2} = \left\| \lambda^m \alpha_1 + \lambda^m \sum_{k=1}^{m-1} (\alpha_{k+1} - \alpha_k) \right\|_{l^2} \\ &\lesssim |\alpha_1| + \left\| \sum_{k=1}^{m-1} \lambda^k \cdot \lambda^{m-k} (\alpha_{m-k+1} - \alpha_{m-k}) \right\|_{l^2} \\ &\leq |\alpha_1| + \sum_{k=1}^{\infty} \lambda^k \|\lambda^m (\alpha_{m+1} - \alpha_m)\|_{l^2} \lesssim \|\alpha\|_{S^\sigma}. \end{aligned}$$

Conversely, for any $\alpha \in \tilde{S}^\sigma$, it is trivial that $\|\alpha\|_{S^\sigma} \lesssim \|\alpha\|_{\tilde{S}^\sigma}$.

(b). Let $\sigma > \frac{d_S}{2}$ and so $\lambda > 1$. For any $\alpha \in S^\sigma$, it is easy to see the limit $c := \lim_{m \rightarrow \infty} \alpha_m = \alpha_1 + \sum_{m=1}^{\infty} (\alpha_{m+1} - \alpha_m)$ exists. Let $\tilde{\alpha}_m = \alpha_m - c$. Then by Minkowski inequality, we have

$$\begin{aligned} \|\tilde{\alpha}\|_{\tilde{S}^\sigma} &= \|\lambda^m \tilde{\alpha}_m\|_{l^2} = \left\| \lambda^m \sum_{k=m}^{\infty} (\alpha_{k+1} - \alpha_k) \right\|_{l^2} \\ &= \left\| \sum_{k=0}^{\infty} \lambda^{-k} \cdot \lambda^{k+m} (\alpha_{k+m+1} - \alpha_{k+m}) \right\|_{l^2} \\ &\leq \sum_{k=0}^{\infty} \lambda^{-k} \|\lambda^m (\alpha_{m+1} - \alpha_m)\|_{l^2} \lesssim \|\alpha\|_{S^\sigma}. \end{aligned}$$

In addition, it is trivial that $|c| \lesssim \|\alpha\|_{S^\sigma}$. Thus we have $|c| + \|\tilde{\alpha}\|_{\tilde{S}^\sigma} \lesssim \|\alpha\|_{S^\sigma}$. The other direction of the estimate is trivial.

(c). Let $\sigma = \frac{d_S}{2}$ and so $\lambda = 1$. Then $\tilde{S}^\sigma = l^2$ and $S^\sigma = \{\alpha : \{(\alpha_{m+1} - \alpha_m)\}_{m \geq 1} \in l^2\}$. The claim is obvious. \square

Before proceeding to the proof of Theorem 3.2, we introduce some more notations here. We will write $\alpha = \{\alpha^p\}_{p \in V_0}$ with each $\alpha^p = (\alpha_1^p, \alpha_2^p, \dots)$ being a sequence. In addition, let $S^\sigma = (S^\sigma)^{V_0}$ and $\tilde{S}^\sigma = (\tilde{S}^\sigma)^{V_0}$, with norms

$$\|\alpha\|_{S^\sigma} = \sum_{p \in V_0} \|\alpha^p\|_{S^\sigma}, \quad \|\alpha\|_{\tilde{S}^\sigma} = \sum_{p \in V_0} \|\alpha^p\|_{\tilde{S}^\sigma}.$$

Since $S^\sigma, \tilde{S}^\sigma$ are stable under complex interpolation (3.2), (3.3), we have the same result for $\mathcal{S}^\sigma, \tilde{\mathcal{S}}^\sigma$, i.e.

$$\begin{aligned} [\mathcal{S}^{\sigma_1}, \mathcal{S}^{\sigma_2}]_\theta &= \mathcal{S}^{(1-\theta)\sigma_1 + \theta\sigma_2}, & \forall \sigma_1, \sigma_2 \in \mathbb{R}, \forall \theta \in (0, 1). \\ [\tilde{\mathcal{S}}^{\sigma_1}, \tilde{\mathcal{S}}^{\sigma_2}]_\theta &= \tilde{\mathcal{S}}^{(1-\theta)\sigma_1 + \theta\sigma_2}, & \forall \sigma_1, \sigma_2 \in \mathbb{R}, \forall \theta \in (0, 1). \end{aligned} \quad (3.4)$$

3.2. Embedding the sequence spaces. In this subsection, we will embed the sequence spaces \mathcal{S}^σ or $\tilde{\mathcal{S}}^\sigma$ into the Sobolev spaces $H^\sigma(K)$, $H_D^\sigma(K)$ or $H_N^\sigma(K)$. In particular, we will introduce a “restriction” map and an “extension” map for different cases.

First, we introduce some notations. For $m \geq 0$, let $\Lambda_m = \Lambda(r^m)$ be as introduced in Definition 2.4 with $r = \min_{i=1}^N r_i$. For each $p \in V_0$, $m \geq 0$, denote

$$\Lambda_{p,m} = \{w \in \Lambda_m : p \in F_w K\}, \text{ and } U_{p,m} = \bigcup_{w \in \Lambda_{p,m}} F_w K.$$

Then $\{U_{p,m}\}_{m \geq 0}$ is a decreasing sequence of neighbourhoods of $\{p\}$. Without loss of generality, we assume

1. $\#\Lambda_{p,1} = \#\pi^{-1}(p)$ for any $p \in V_0$;
2. $U_{p,1} \cap U_{q,1} = \emptyset$ for any $p, q \in V_0$;
3. $(U_{p,m})^c \cap U_{p,m+1} = \emptyset$ for any $p \in V_0$ and $m \geq 1$,

otherwise we replace r by a sufficiently small number.

Lemma 3.5. *For each $p \in V_0$, there exist two functions ϕ_p and ψ_p in \mathcal{H}_0 such that for any $h \in \mathcal{H}_0$, we have*

$$\int_K \phi_p h d\mu = h(p), \text{ and } \int_K \psi_p h d\mu = \partial_n h(p).$$

Proof. Observing that $h \rightarrow h(p)$ and $h \rightarrow \partial_n h(p)$ are functionals on \mathcal{H}_0 , the lemma follows immediately from Riesz representation theorem. \square

Definition 3.6. *For $p \in V_0$, $m \geq 1$, let $\phi_{p,m}$ and $\psi_{p,m}$ be two functions supported in $U_{p,m}$ such that*

$$\phi_{p,m} = \sum_{w \in \Lambda_{p,m}} r_w^{-d_H} \phi_{F_w^{-1}p} \circ F_w^{-1} \text{ and } \psi_{p,m} = \sum_{w \in \Lambda_{p,m}} r_w^{-1-d_H} \psi_{F_w^{-1}p} \circ F_w^{-1}.$$

Furthermore, for $f \in L^2(K)$, define

- (a). $R_v^p f = \left\{ \int_K \phi_{p,m} f d\mu \right\}_{m \geq 1}$ and write $R_v f = \{R_v^p f\}_{p \in V_0}$;
- (b). $R_n^p f = \left\{ \int_K \psi_{p,m} f d\mu \right\}_{m \geq 1}$ and write $R_n f = \{R_n^p f\}_{p \in V_0}$.

Remark. The operators R_v and R_n will play the role of the “restriction” map. In particular, $\lim_{m \rightarrow \infty} (R_v^p f)_m = \#\pi^{-1}(p)f(p)$ and $\lim_{m \rightarrow \infty} (R_n^p f)_m = \partial_n f(p)$ for any $f \in H^2(K)$ and $p \in V_0$.

In fact, since $H^2(K) \subset C(K)$ ($H^2(K) \subset \text{dom}\mathcal{E} \subset C(K)$) and any function f in $H^2(K)$ admits normal derivatives at V_0 (see Lemma 3.7.5 of [24]), the remark follows from Lemma 3.5, through a routine argument by splitting f near p into a sum of a harmonic function and a function with Dirichlet boundary condition.

Lemma 3.7. *Let $0 \leq \sigma \leq 2$.*

- (a). *The map R_v is bounded from $H^\sigma(K)$ to \mathcal{S}^σ , and also from $H_D^\sigma(K)$ to $\tilde{\mathcal{S}}^\sigma$.*
- (b). *The map R_n is bounded from $H^\sigma(K)$ to $\mathcal{S}^{\sigma-2/d_w}$, and also from $H_N^\sigma(K)$ to $\tilde{\mathcal{S}}^{\sigma-2/d_w}$.*

Proof. We only prove (a), and the proof of (b) is essentially the same. First we show $R_v : H^\sigma(K) \rightarrow \mathcal{S}^\sigma$ is bounded. By complex interpolation, and using (3.4), we only need to show it for $\sigma = 0$ and $\sigma = 2$.

For $\sigma = 0$, it follows from the estimates that for any $f \in L^2(K)$ and any $p \in V_0$, we have

$$\begin{aligned}
\|R_v^p f\|_{\mathcal{S}^0} &\asymp \|R_v^p f\|_{\tilde{\mathcal{S}}^0} = \|r^{md_H/2} \int_{U_{p,m}} \phi_{p,m} f d\mu\|_{l^2} \lesssim \|r^{-md_H/2} \int_{U_{p,m}} |f| d\mu\|_{l^2} \\
&\lesssim \|r^{-md_H/2} \sum_{k=m}^{\infty} r^{kd_H/2} \|f\|_{L^2(U_{p,k} \setminus U_{p,k+1})}\|_{l^2} \\
&= \|r^{-md_H/2} \sum_{k=0}^{\infty} r^{(m+k)d_H/2} \|f\|_{L^2(U_{p,m+k} \setminus U_{p,m+k+1})}\|_{l^2} \\
&\leq \sum_{k=0}^{\infty} r^{kd_H/2} \| \|f\|_{L^2(U_{p,m+k} \setminus U_{p,m+k+1})}\|_{l^2} \lesssim \|f\|_{L^2(K)}.
\end{aligned} \tag{3.5}$$

where we use Lemma 3.4 (a) in the first estimate, and use Cauchy-Schwartz inequality, Minkowski inequality in the remaining estimates.

For $\sigma = 2$, let $\tilde{\phi}_{p,m} = \phi_{p,m+1} - \phi_{p,m}$, $\forall p \in V_0, m \geq 1$. For each $w \in \Lambda_{p,m}$, immediately we have $\int_{F_w K} \tilde{\phi}_{p,m} h d\mu = 0$, for each h harmonic in $F_w K$. As a result, by using Gauss-Green's formula on each $F_w K$, for any $f \in H^2(K)$, we have

$$\int_{F_w K} \tilde{\phi}_{p,m} f d\mu = \int_{F_w K} \tilde{\phi}_{p,m} (f - h) d\mu = - \int_{F_w K} G_w \tilde{\phi}_{p,m} \cdot \Delta f d\mu,$$

where h is harmonic in $F_w K$ with $h|_{F_w V_0} = f|_{F_w V_0}$, and G_w is the local Green's function associated with $F_w K$. Define

$$\phi'_{p,m} = - \sum_{w \in \Lambda_{p,m}} G_w \tilde{\phi}_{p,m}.$$

Then, it is easy to see that

$$(R_v^p f)_{m+1} - (R_v^p f)_m = \int_{U_{p,m}} \phi'_{p,m} \cdot \Delta f d\mu.$$

In addition, we have the estimate $\|\phi'_{p,m}\|_{L^\infty(U_{p,m})} \lesssim r^m$. The result for $\sigma = 2$ then follows a by similar estimate as (3.5).

For the boundedness of $R_v : H_D^\sigma(K) \rightarrow \tilde{\mathcal{S}}^\sigma$, we still use the complex interpolation for $0 \leq \sigma \leq 2$. For $\sigma = 0$, it follows immediately since $H_D^0(K) = L^2(K)$. For $\sigma = 2$, we only need to notice that for any $p \in V_0$, $f \in H_D^2(K)$, we always have $\lim_{m \rightarrow \infty} r^{-md_H/2} \int_{U_{p,m}} |f| d\mu = 0$, so that $R_v f \in \tilde{\mathcal{S}}^2$ and $\|R_v f\|_{\mathcal{S}^2} \asymp \|R_v f\|_{\tilde{\mathcal{S}}^2}$ by Lemma 3.4 (b). \square

In the next lemma, we construct maps E_v, E_n that play the role of “extension” map.

Lemma 3.8. *Let $0 \leq \sigma \leq 2$.*

- (a). *There exists a bounded map $E_v : \mathcal{S}^\sigma \rightarrow H^\sigma(K)$ satisfying $R_v \circ E_v = id$.*
- (b). *There exists a bounded map $E_n : \mathcal{S}^{\sigma-2/dw} \rightarrow H^\sigma(K)$ satisfying $R_n \circ E_n = id$.*

Proof. (a). For $p \in V_0$ and $m \geq 1$, we choose a function $g_{p,m} \in H^2(K)$ such that

$$g_{p,m}|_{U_{p,m+1}} = (\#\pi^{-1}(p))^{-1}, \quad g_{p,m}|_{K \setminus U_{p,m}} = 0,$$

and

$$(R_v^p g_{p,m})_k = \begin{cases} 0, & \text{for } k \leq m, \\ 1, & \text{for } k \geq m+1. \end{cases}$$

In addition, for each $p \in V_0$, we can guarantee

$$\sup_{m \geq 1} r^{-m(d_H/2+1)} \|\Delta g_{p,m}\|_{L^2(K)} < \infty, \quad \sup_{m \geq 1} \|g_{p,m}\|_{L^\infty(K)} < \infty.$$

In fact, we can construct $g_{p,m}|_{F_w K \setminus U_{p,m+1}}$ for each $w \in \Lambda_{p,m}$ respectively, and only finitely many different cases of $F_w K \setminus U_{p,m+1}$ (up to some contraction mapping F_v) will occur. The estimate for $\|g_{p,m}\|_{L^\infty(K)}$ is immediate, and the estimate for $\|\Delta g_{p,m}\|_{L^2(K)}$ follows from the scaling property of Δ .

For each $\alpha \in \mathcal{S}^\sigma$, define

$$E_v \alpha = h + \sum_{p \in V_0} \sum_{m=1}^{\infty} (\alpha_{m+1}^p - \alpha_m^p) g_{p,m},$$

where h is a harmonic function with boundary values $h(p) = (\#\pi^{-1}(p))^{-1} \alpha_1^p$, $\forall p \in V_0$.

We will show that $E_v : \mathcal{S}^\sigma \rightarrow L_\sigma^2(K)$ is bounded. By complex interpolation, and by using (3.4), it is enough to show it for $\sigma = 0$ and $\sigma = 2$. For $\sigma = 0$, by using Minkowski inequality, for each $\alpha \in \mathcal{S}^0$, we have the estimate

$$\begin{aligned} \|E_v \alpha\|_{L^2(K)} &\leq \sum_{p \in V_0} \left\| \sum_{k=1}^{\infty} (\alpha_{k+1}^p - \alpha_k^p) g_{p,k} \right\|_{L^2(U_{p,1})} + \|\alpha\|_{\mathcal{S}^0} \\ &= \sum_{p \in V_0} \left\| \left\| \sum_{k=1}^m (\alpha_{k+1}^p - \alpha_k^p) g_{p,k} \right\|_{L^2(U_{p,m} \setminus U_{p,m+1})} \right\|_{l^2} + \|\alpha\|_{\mathcal{S}^0} \\ &\lesssim \sum_{p \in V_0} \left\| r^{md_H/2} \sum_{k=1}^m |\alpha_{k+1}^p - \alpha_k^p| \right\|_{l^2} + \|\alpha\|_{\mathcal{S}^0} \\ &= \sum_{p \in V_0} \left\| \sum_{k=0}^{m-1} r^{kd_H/2} \cdot r^{(m-k)d_H/2} |\alpha_{m-k+1}^p - \alpha_{m-k}^p| \right\|_{l^2} + \|\alpha\|_{\mathcal{S}^0} \\ &\lesssim \|\alpha\|_{\mathcal{S}^0}. \end{aligned}$$

For $\sigma = 2$, the proof is easy, noticing each $\Delta g_{p,m}$ is locally supported on $U_{p,m} \setminus U_{p,m+1}$.

Lastly, it is direct to check that $R_v(E_v \alpha) = \alpha$ on \mathcal{S}^0 .

The proof of (b) is essentially the same. The main difference is that we construct $g_{p,m}$ to be harmonic in each cell of $U_{p,m+1}$ with desired normal derivative at p . We omit the details. \square

As an immediate consequence of Lemma 3.8, and by the remark before Lemma 3.7, we have the following lemma.

Lemma 3.9. *Let $0 \leq \sigma \leq 2$, and E_v, E_n be the map defined in Lemma 3.8. Then*

- (a). E_v is bounded from $\mathcal{S}^\sigma \rightarrow H_D^\sigma(K)$.
- (b). E_n is bounded from $\tilde{\mathcal{S}}^{\sigma-2/d_W}$ to $H_N^\sigma(K)$.

3.3. Proof of the comparison theorem. In this part, we come to the proof of Theorem 3.2. We need to use the following simple fact, which can be easily derived from the property of interpolation functors.

Lemma 3.10. *Let (Z_1, Z_2) be an interpolation couple with $Z_1 = X_1 \oplus Y_1$, $Z_2 = X_2 \oplus Y_2$, and $(X_1 + X_2) \cap (Y_1 + Y_2) = \{0\}$. Then we have*

$$[Z_1, Z_2]_\theta = [X_1, X_2]_\theta \oplus [Y_1, Y_2]_\theta, \quad \forall 0 < \theta < 1.$$

The following lemma concludes what we have got in the last two subsections.

Lemma 3.11. *Let $0 \leq \sigma \leq 2$. Define*

$$\ker_\sigma R_v = \{f \in H^\sigma(K) : R_v f = 0\} \text{ and } \ker_\sigma R_n = \{f \in H^\sigma(K) : R_n f = 0\}.$$

Then we have

- (a). $H^\sigma(K) = E_v \mathcal{S}^\sigma \oplus \ker_\sigma R_v$ and $H_D^\sigma(K) = E_v \tilde{\mathcal{S}}^\sigma \oplus \ker_\sigma R_v$.
- (b). $H^\sigma(K) = E_n \mathcal{S}^{\sigma-2/dw} \oplus \ker_\sigma R_n$ and $H_N^\sigma(K) = E_n \tilde{\mathcal{S}}^{\sigma-2/dw} \oplus \ker_\sigma R_n$.

Proof. (a). The first identity is obvious by Lemma 3.7 and Lemma 3.8. In addition, we can see that

$$\ker_\sigma R_v = [\ker_0 R_v, \ker_2 R_v]_{\sigma/2}, \quad \forall 0 < \sigma < 2, \quad (3.6)$$

by applying Lemma 3.10.

We can also see from the first identity that $\ker_\sigma R_v = \{f \in H_D^\sigma(K) : R_v f = 0\}$ for $\sigma = 0, 2$ using the remark before Lemma 3.7. Applying Lemma 3.7 and Lemma 3.9, we then have the second identity holds for $\sigma = 0, 2$. Since $H_D^\sigma(K)$ and $\tilde{\mathcal{S}}^\sigma$ are stable under complex interpolation, we have

$$H_D^\sigma(K) = E_v \tilde{\mathcal{S}}^\sigma \oplus [\ker_0 R_v, \ker_2 R_v]_{\sigma/2},$$

by applying Lemma 3.10. The second identify then follows from (3.6) immediately.

The proof of (b) is the same. □

Proof of Theorem 3.2. (a). For $0 \leq \sigma \leq 2$, the claims are easy consequences of Lemma 3.11. For $0 \leq \sigma < \frac{d_S}{2}$, the result follows from Lemma 3.11 and Lemma 3.4 (a). For $\sigma = \frac{d_S}{2}$, the result follows from Lemma 3.11 and Lemma 3.4 (c). For $\frac{d_S}{2} < \sigma \leq 2$, one have $\mathcal{H}_0 \cap H_D^\sigma(K) = \{0\}$ since obviously $R_v \mathcal{H}_0 \cap R_v H_D^\sigma(K) = \{0\}$ and $\ker_\sigma R_v \cap \mathcal{H}_0 = \{0\}$. The result follows from Lemma 3.11, Lemma 3.4 (b) and codimension counting.

The result for $2k \leq \sigma \leq 2k + 2$ follows from the fact that

$$H_D^\sigma(K) = G^k H_D^{\sigma-2k}(K), \text{ and } H^\sigma(K) = \mathcal{H}_{k-1} \oplus G^k H^{\sigma-2k}(K). \quad (3.7)$$

To see the second equality, we follow a similar proof as Proposition 3.1 for $\sigma = 2k, 2k + 2$ and then apply Lemma 3.10.

(b). For $0 \leq \sigma \leq 2$, the proof is the same as (a). For $2k \leq \sigma \leq 2k + 2$, similar to (a), we have

$$H_N^\sigma(K) = \text{constants} \oplus G_N^k H_N^{\sigma-2k}(K), \text{ and } H^\sigma(K) = \mathcal{H}'_{k-1} \oplus (\text{constants} \oplus G_N^k H^{\sigma-2k}(K)).$$

□

4. THE ATOMIC DECOMPOSITION AND OTHER BESOV TYPE CHARACTERIZATIONS

It is of interest to see what kind of role the critical orders in Theorem 3.2 play. It is well-known that $\frac{d_S}{2}$ is a critical order of continuity of functions in Sobolev spaces $H^\sigma(K)$. Also, we expect that $2 - \frac{d_S}{2}$ is a critical order concerning Hölder continuity of functions in $H^\sigma(K)$. The forthcoming atomic decomposition theorem will also provide an explanation.

4.1. The atomic decomposition. We denote χ_A for the characteristic function on a set A contained in K , i.e. $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.

We denote ψ_x for the tent function at a point $x \in V_* \setminus V_0$. To be more precise, for $m \in \mathbb{N}$ and $x \in V_m \setminus V_{m-1}$, we define $\psi_x(y) = 1$ if $y = x$, $\psi_x(y) = 0$ if $y \in V_m \setminus \{x\}$, and extend ψ_x to be harmonic in each cell $F_w K$ with $w \in W_m$.

Theorem 4.1. (a). For $k \geq 0$ and $\sigma \in (2k - \frac{d_S}{2}, 2k + \frac{d_S}{2}) \cap \mathbb{R}_+$, the series

$$f = h + G^k C + \sum_{w \in W_*} \sum_{i=1}^N a_{wi} G^k \chi_{F_{wi} K}, \quad (4.1)$$

with $h \in \mathcal{H}_{k-1}$, $C \in \mathbb{R}$, $a_{wi} \in \mathbb{R}$, $\sum_{i=1}^N r_i^{d_H} a_{wi} = 0$, is in $H^\sigma(K)$ if and only if

$$\sum_{w \in W_*} r_w^{d_H - (\sigma - 2k)d_W} \sum_{i=1}^N |a_{wi}|^2 < \infty,$$

with $\|f\|_{H^\sigma(K)} \asymp (\|h\|_{L^2(K)}^2 + |C|^2 + \sum_{w \in W_*} r_w^{d_H - (\sigma - 2k)d_W} \sum_{i=1}^N |a_{wi}|^2)^{1/2}$. In addition, each f in $H^\sigma(K)$ admits a unique expansion of this form.

(b). For $k \geq 0$ and $\sigma \in (2k + \frac{d_S}{2}, 2k + 2 - \frac{d_S}{2}) \cap \mathbb{R}_+$, the series

$$f = h + \sum_{x \in V_* \setminus V_0} c_x G^k \psi_x, \quad (4.2)$$

with $h \in \mathcal{H}_k$ and $c_x \in \mathbb{R}$, is in $H^\sigma(K)$ if and only if

$$\sum_{w \in W_*} r_w^{d_H - (\sigma - 2k)d_W} \sum_{x \in V_1 \setminus V_0} |c_{F_w x}|^2 < \infty,$$

with $\|f\|_{H^\sigma(K)} \asymp (\|h\|_{L^2(K)}^2 + \sum_{w \in W_*} r_w^{d_H - (\sigma - 2k)d_W} \sum_{x \in V_1 \setminus V_0} |c_{F_w x}|^2)^{1/2}$. In addition, each f in $H^\sigma(K)$ admits a unique expansion of this form.

Remark 1. Comparing Theorem 4.1 with Theorem 3.2, we can immediately get a similar atomic decomposition theorem for $H_D^\sigma(K)$ with $\sigma \geq 0$. The statement is almost the same with the multiharmonic function term h removed from the expansions in (4.1) and (4.2).

Remark 2. There are some other types of atomic decomposition theorems. See Theorem 6.7 and Theorem 8.4 in later sections.

We will prove Theorem 4.1 in the remaining part of this paper. Due to formula (3.7), we only need to consider orders $0 \leq \sigma < 2$. We will split the proof into two main parts: $0 \leq \sigma < 1$ and $1 \leq \sigma < 2$. The case $1 \leq \sigma < 2$ is more or less similar to the case $0 \leq \sigma < 1$, and so some details will be omitted in the second part.

4.2. Besov type characterizations. At the mean time, we will provide some other types of characterizations of $H^\sigma(K)$. Again, we will postpone the proof in later sections. Throughout this paper, we write $\lambda = \lambda(\sigma) = r^{(d_H - \sigma d_w)/2}$ as in Section 3.

We follow Strichartz [35] to define the following Besov type spaces $\Lambda_\sigma^{2,2}(K)$ and $\tilde{\Lambda}_\sigma^{2,2}(K)$, based on discrete differences. The spaces are defined with different differences for $\frac{d_S}{2} < \sigma < 1$ and $\frac{d_S}{2} < \sigma < 2$ separately.

For $\frac{d_S}{2} < \sigma < 1$, we consider the following spaces.

Definition 4.2. For $m \geq 0$, we write $V_{\Lambda_m} = \bigcup_{w \in \Lambda_m} F_w V_0$ for short.

(a). For $m \geq 0$, call the graph $G_{v,m} = (V_{\Lambda_m}, E_{v,m})$ with vertice set V_{Λ_m} and edge set $E_{v,m}$ a level- m vertex graph, where $\{x, y\} \in E_{v,m}$ if and only if there exists $w \in \Lambda_m$ and $p, q \in V_0$ such that $x = F_w p, y = F_w q$.

(b). For $m \geq 0$, define the difference operator $\nabla_m : C(K) \rightarrow l(E_{v,m})$ as

$$\nabla_m f(\{x, y\}) = |f(x) - f(y)|, \quad \forall f \in C(K) \text{ and } \{x, y\} \in E_{v,m}.$$

Definition 4.3. For $\sigma > \frac{d_S}{2}$, define

$$\Lambda_\sigma^{2,2}(K) = \{f \in C(K) : \sum_{m=0}^{\infty} \lambda^{2m} \|\nabla_m f\|_{l^2(E_{v,m})}^2 < \infty\},$$

with norm $\|f\|_{\Lambda_\sigma^{2,2}(K)} = (\|f\|_{L^2(K)}^2 + \sum_{m=0}^{\infty} \lambda^{2m} \|\nabla_m f\|_{l^2(E_{v,m})}^2)^{1/2}$.

One can easily check that $r^{-m} \|\nabla_m f\|_{l^2(E_{v,m})}^2 \asymp \mathcal{E}_{\Lambda_m}(f)$, where $\mathcal{E}_{\Lambda_m}(f)$ is the discrete energy approximating \mathcal{E} to be defined in the following Definition 4.4.

Definition 4.4. For $m \geq 0$, denote the (discrete) Dirichlet form on V_{Λ_m} by

$$\mathcal{E}_{\Lambda_m}(f, g) = \sum_{w \in \Lambda_m} r_w^{-1} \mathcal{E}_0(f \circ F_w, g \circ F_w), \quad \forall f, g \in l(V_{\Lambda_m}),$$

and write its corresponding graph Laplacian as $H_{\Lambda_m} : l(V_{\Lambda_m}) \rightarrow l(V_{\Lambda_m})$, i.e.,

$$\mathcal{E}_{\Lambda_m}(f, g) = -(f, H_{\Lambda_m} g), \quad \forall f, g \in l(V_{\Lambda_m}).$$

We write $\mathcal{E}_{\Lambda_m}(f) = \mathcal{E}_{\Lambda_m}(f, f)$ for short.

The following Lemma 4.5 verifies that \mathcal{E}_{Λ_m} is a sequence of discrete energies approximating \mathcal{E} , similarly to (2.1).

Lemma 4.5. The following properties hold for any $f \in C(K)$.

- (a). $\mathcal{E}_{\Lambda_0}(f) \leq \mathcal{E}_{\Lambda_1}(f) \leq \mathcal{E}_{\Lambda_2}(f) \leq \dots$.
- (b). $\mathcal{E}(f) = \lim_{m \rightarrow \infty} \mathcal{E}_{\Lambda_m}(f)$.

Proof. Recall that $\Sigma = \{1, 2, \dots, N\}^{\mathbb{N}}$ is the shift space, and we write $\Sigma_w = \{\omega \in \Sigma, [\omega]_m = w\}$ for any $w \in W_m$ and $m \geq 0$. We call $\Lambda \subset W_*$ a *partition* of Σ if $\Sigma = \bigcup_{w \in \Lambda} \Sigma_w$, and $\Sigma_w \cap \Sigma_{w'} = \emptyset$ for any $w \neq w'$ in Λ . For any partition Λ , one can define

$$\mathcal{E}_\Lambda(f, g) = \sum_{w \in \Lambda} r_w^{-1} \mathcal{E}_0(f \circ F_w, g \circ F_w), \quad \forall f, g \in l(V_\Lambda),$$

where $V_\Lambda = \bigcup_{w \in \Lambda} F_w V_0$. By abuse of the notation, we simply write $\mathcal{E}_\Lambda(f) = \mathcal{E}_\Lambda(f|_{V_\Lambda}, f|_{V_\Lambda})$ for any $f \in C(K)$.

We say a partition Λ' is a *refinement* of Λ if for any $w' \in \Lambda'$ there exists $w \in \Lambda$ so that $\Sigma_{w'} \subset \Sigma_w$. One may iteratively define $\Lambda^{(0)} = \Lambda$, and $\Lambda^{(l+1)} = (\Lambda^{(l)} \cap \Lambda') \cup \{w_i : w \in \Lambda^{(l)} \setminus \Lambda', 1 \leq i \leq N\}$ for $l \geq 0$. Then,

$$\begin{aligned} \mathcal{E}_{\Lambda^{(l)}}(f) &= \sum_{w \in \Lambda^{(l)}} r_w^{-1} \mathcal{E}_0(f \circ F_w) \\ &\leq \sum_{w \in \Lambda^{(l)} \cap \Lambda'} r_w^{-1} \mathcal{E}_0(f \circ F_w) + \sum_{w \in \Lambda^{(l)} \setminus \Lambda'} r_w^{-1} \mathcal{E}_1(f \circ F_w) \\ &= \sum_{w \in \Lambda^{(l+1)}} r_w^{-1} \mathcal{E}_0(f \circ F_w) = \mathcal{E}_{\Lambda^{(l+1)}}(f). \end{aligned}$$

Given $\#\Lambda' < \infty$, there is $l \geq 0$ such that $\Lambda^{(l)} = \Lambda'$ so that we have

$$\mathcal{E}_{\Lambda}(f) = \mathcal{E}_{\Lambda^{(0)}}(f) \leq \mathcal{E}_{\Lambda^{(1)}}(f) \leq \cdots \leq \mathcal{E}_{\Lambda^{(l)}}(f) = \mathcal{E}_{\Lambda'}(f).$$

Thus, for any pair of partitions Λ', Λ such that Λ' is a refinement of Λ , we have $\mathcal{E}_{\Lambda}(f) \leq \mathcal{E}_{\Lambda'}(f)$.

(a). Clearly, Λ_{m+1} is a refinement of Λ_m , so (a) follows immediately.

(b). Clearly, Λ_m is a refinement of W_m for any $m \geq 0$. On the other hand, one can always find large N_m so that W_{N_m} is a refinement of Λ_m . Thus, $\mathcal{E}_m(f) \leq \mathcal{E}_{\Lambda_m}(f) \leq \mathcal{E}_{N_m}(f)$. (b) then follows from (2.1). \square

Remark. We point out that $\Lambda_{\sigma}^{2,2}(K) = \text{constants}$ when $\sigma \geq 1$, since then

$$\mathcal{E}(f) = \lim_{m \rightarrow \infty} \mathcal{E}_{\Lambda_m}(f) \lesssim \limsup_{m \rightarrow \infty} r^{-m} \|\nabla_m f\|_{l^2(E_{v,m})}^2 = 0, \quad \forall f \in \Lambda_{\sigma}^{2,2}(K).$$

For $\frac{d_S}{2} < \sigma < 2$, we consider the following spaces.

Definition 4.6. For $\sigma > \frac{d_S}{2}$, define

$$\tilde{\Lambda}_{\sigma}^{2,2}(K) = \{f \in C(K) : \sum_{m=1}^{\infty} r^{2m} \lambda^{2m} \|H_{\Lambda_m} f\|_{l^2(V_{\Lambda_m} \setminus V_0)}^2 < \infty\},$$

with norm $\|f\|_{\tilde{\Lambda}_{\sigma}^{2,2}(K)} = (\|f\|_{L^2(K)}^2 + \sum_{m=1}^{\infty} r^{2m} \lambda^{2m} \|H_{\Lambda_m} f\|_{l^2(V_{\Lambda_m} \setminus V_0)}^2)^{1/2}$.

We will prove the following theorem.

Theorem 4.7. (a). For $\frac{d_S}{2} < \sigma < 1$, we have $H^{\sigma}(K) = \Lambda_{\sigma}^{2,2}(K)$ with $\|\cdot\|_{H^{\sigma}(K)} \asymp \|\cdot\|_{\Lambda_{\sigma}^{2,2}(K)}$.

(b). For $\frac{d_S}{2} < \sigma < 2$, we have $H^{\sigma}(K) = \tilde{\Lambda}_{\sigma}^{2,2}(K)$ with $\|\cdot\|_{H^{\sigma}(K)} \asymp \|\cdot\|_{\tilde{\Lambda}_{\sigma}^{2,2}(K)}$.

Another class of Besov type spaces on fractals, denoted by $B_{\sigma}^{2,2}(K)$, given in Definition 4.8 below, has been widely studied in connection with heat kernel and Dirichlet forms, see [13, 15, 17, 18, 20, 30]. Also see [1, 2] for analogous spaces with slight differences. It was shown in [18] on general metric measure spaces that $H_N^{\sigma}(K)$ are identical with $B_{\sigma}^{2,2}(K)$ when $0 < \sigma < 1$ under the assumption of nice heat kernel estimates. As an application of our results, we will give a purely analytical way to prove this result on p.c.f. self-similar sets satisfying our assumptions, without using the heat kernel estimates. This could also be proved by combining results of [14] and [18].

Definition 4.8. For $\sigma > 0$ and $f \in L^2(K)$, denote

$$[f]_{B_\sigma^{2,2}(K)} = \left(\int_0^1 \frac{dt}{t} \int_K \int_{B(x,t)} t^{-d_H - \sigma d_W} |f(x) - f(y)|^2 d\mu(y) d\mu(x) \right)^{1/2}.$$

Define the Besov space $B_\sigma^{2,2}(K)$ to be

$$B_\sigma^{2,2}(K) = \{f \in L^2(K) : [f]_{B_\sigma^{2,2}(K)} < \infty\}$$

with the norm

$$\|f\|_{B_\sigma^{2,2}(K)} = \|f\|_{L^2(K)} + [f]_{B_\sigma^{2,2}(K)}.$$

Theorem 4.9. For $0 < \sigma < 1$, we have $H^\sigma(K) = B_\sigma^{2,2}(K)$ with $\|\cdot\|_{H^\sigma(K)} \asymp \|\cdot\|_{B_\sigma^{2,2}(K)}$.

Lastly, comparing the spaces $\Lambda_\sigma^{2,2}(K)$ in terms of vertex graph approximations of K , we will provide some other Besov type characterizations of $H^\sigma(K)$ based on the cell graph approximations of K . See Theorem 6.1, Corollary 6.5 and Theorem 6.7.

5. HAAR SERIES EXPANSION AND CELL GRAPH REPRESENTATION

For $m \geq 1$, Λ_m provides a nested partition of K consisting of cells with comparable diameters. Accordingly, there is a natural cell graph associated with Λ_m inherited from the topology of K .

Definition 5.1. For $m \geq 1$, call the graph $G_{c,m} = (\Lambda_m, E_{c,m})$ with vertex set Λ_m and edge set $E_{c,m}$ a level- m cell graph, where $\{w, w'\} \in E_{c,m}$ if and only if $F_w K \cap F_{w'} K \neq \emptyset$.

In [34], motivated by the work of Kusuoka and Zhou [27], Strichartz introduced the notions of *cell graphs* and *cell graph energies* for the *Sierpinski gasket* \mathcal{SG} , and provided an equivalent definition of Laplacians on \mathcal{SG} , instead of using vertex graphs and vertex graph energies as we usually did in fractal analysis. See [31, 36] for some interesting works using related considerations.

In this section, basing on cell decomposition of functions, we will introduce a class of Besov type function spaces $\Gamma^\sigma(K)$ on K , which will be proved later to identify with $H^\sigma(K)$ when $0 \leq \sigma < 1$. Recall that for $\sigma \geq 0$, we always write $\lambda = \lambda(\sigma) = r^{(d_H - \sigma d_W)/2}$ for short, where $r = \min_{i=1}^N r_i$ and $d_W = 1 + d_H$.

Firstly, let's look at decompositions of L^2 functions on K by only using the nested structure provided by $\{\Lambda_m\}_{m \geq 0}$.

5.1. Haar series expansion.

Definition 5.2. Let $f \in L^2(K)$.

(a). For any $w \in W_*$, define the average value of f on $F_w K$ by

$$A_w(f) = \frac{1}{\mu(F_w K)} \int_{F_w K} f d\mu.$$

In particular, $A_\emptyset(f) = \int_K f d\mu$.

(b). For $m \geq 1$, define the space of m -Haar functions

$$\tilde{J}_m = \{\tilde{f}_m = \sum_{w \in \Lambda_m} c_w \chi_{F_w K} : c_w \in \mathbb{R}, A_{w'}(\tilde{f}_m) = 0, \forall w' \in \Lambda_{m-1}\}.$$

It is easy to see that as subspaces of $L^2(K)$, $\tilde{J}_m \perp \tilde{J}_{m'}, \forall m \neq m'$, and

$$L^2(K) = \text{constants} \oplus \left(\bigoplus_{m=1}^{\infty} \tilde{J}_m \right),$$

where we use the standard inner product $(f, g) = \int_K f \cdot g d\mu$ for $f, g \in L^2(K)$. Thus for any $f \in L^2(K)$, there is a unique expansion

$$f = C + \sum_{m=1}^{\infty} \tilde{f}_m, \text{ with } \tilde{f}_m \in \tilde{J}_m.$$

The following subspaces of $L^2(K)$ are defined with weighted Haar series.

Definition 5.3. For $\sigma \geq 0$, define

$$\begin{aligned} \tilde{\Gamma}^\sigma(K) = \{f \in L^2(K) : f = C + \sum_{m=1}^{\infty} \tilde{f}_m, \text{ with } \tilde{f}_m \in \tilde{J}_m, \\ \text{satisfying } \sum_{m=1}^{\infty} r^{-md_H} \lambda^{2m} \|\tilde{f}_m\|_{L^2(K)}^2 < \infty\}, \end{aligned}$$

with the norm defined by

$$\|f\|_{\tilde{\Gamma}^\sigma(K)} = \left(|C|^2 + \sum_{m=1}^{\infty} r^{-md_H} \lambda^{2m} \|\tilde{f}_m\|_{L^2(K)}^2 \right)^{1/2}.$$

In particular, $\tilde{\Gamma}^0(K) = L^2(K)$.

5.2. Cell graph representation. Haar series expansion of functions gives an easy description of functions in $L^2(K)$. However, the drawback is that the information related to the topology of K is lost. To save this, we need to use the cell graphs $\{G_{c,m}\}_{m \geq 1}$.

In particular, we are interested in the cell graph energies.

Definition 5.4. Let $m \geq 1$.

(a). Define the cell graph difference operator $D_m : L^2(K) \rightarrow l(E_{c,m})$ as

$$D_m f(\{w, w'\}) = |A_w(f) - A_{w'}(f)|, \quad \forall f \in L^2(K) \text{ and } \{w, w'\} \in E_{c,m}.$$

(b). For $f \in L^2(K)$, define its level- m cell graph energy as $\|D_m f\|_{l^2(E_{c,m})}^2$.

Based on the cell graph energies, we define the following Besov type spaces $\Gamma^\sigma(K)$ for $\sigma \geq 0$.

Definition 5.5. For $\sigma \geq 0$, define the space

$$\Gamma^\sigma(K) = \{f \in L^2(K) : \sum_{m=1}^{\infty} \lambda^{2m} \|D_m f\|_{l^2(E_{c,m})}^2 < \infty\},$$

with the norm defined by

$$\|f\|_{\Gamma^\sigma(K)} = \left(\|f\|_{L^2(K)}^2 + \sum_{m=1}^{\infty} \lambda^{2m} \|D_m f\|_{l^2(E_{c,m})}^2 \right)^{1/2}.$$

There is a close relation between $\Gamma^\sigma(K)$ and $\tilde{\Gamma}^\sigma(K)$. To illustrate this, we introduce following notations.

Definition 5.6. (a). For each $w \in W_*$, define $W_w = \{w' \in W_* : F_{w'}K \subset F_wK\}$.

(b). For $m \geq 1$, denote $\tilde{G}_{c,m} = (\Lambda_m, \tilde{E}_{c,m})$, where $\{w, w'\} \in \tilde{E}_{c,m}$ if and only if $\{w, w'\} \in E_{c,m}$ and both w, w' belong to some $W_{w''}$ with $w'' \in \Lambda_{m-1}$.

(c). For $m \geq 1$, define the operator $\tilde{D}_m : L^2(K) \rightarrow l(\tilde{E}_{c,m})$ as

$$\tilde{D}_m f(\{w, w'\}) = |A_w(f) - A_{w'}(f)|, \quad \forall f \in L^2(K) \text{ and } \{w, w'\} \in \tilde{E}_{c,m}.$$

This definition is similar to Definition 5.1 and 5.4. Clearly, $\tilde{E}_{c,m} \subset E_{c,m}$ and $\|\tilde{D}_m f\|_{l^2(\tilde{E}_{c,m})} \leq \|D_m f\|_{l^2(E_{c,m})}, \forall f \in L^2(K)$. Analogous to Definition 5.5, we have the following characterization of $\tilde{\Gamma}^\sigma(K)$ for $\sigma \geq 0$.

Proposition 5.7. For $\sigma \geq 0$, we have

$$\tilde{\Gamma}^\sigma(K) = \{f \in L^2(K) : \sum_{m=1}^{\infty} \lambda^{2m} \|\tilde{D}_m f\|_{l^2(\tilde{E}_{c,m})}^2 < \infty\},$$

with

$$\|f\|_{\tilde{\Gamma}^\sigma(K)} \asymp \left(\|f\|_{L^2(K)}^2 + \sum_{m=1}^{\infty} \lambda^{2m} \|\tilde{D}_m f\|_{l^2(\tilde{E}_{c,m})}^2 \right)^{1/2}.$$

Proof. It is direct to see that

$$\tilde{D}_m \tilde{f}_l = 0, \text{ for any } l \neq m, \tilde{f}_l \in \tilde{J}_l,$$

from Definition 5.2 and 5.6. Thus, for $f = C + \sum_{l=1}^{\infty} \tilde{f}_l$ with $\tilde{f}_l \in \tilde{J}_l$, we have

$$\tilde{D}_m f = \tilde{D}_m C + \sum_{l=1}^{\infty} \tilde{D}_m \tilde{f}_l = \tilde{D}_m \tilde{f}_m.$$

The proposition follows from the observation that $\|\tilde{D}_m \tilde{f}_m\|_{l^2(\tilde{E}_{c,m})} \asymp r^{-md_H/2} \|\tilde{f}_m\|_{L^2(K)}$. \square

Immediately, by Proposition 5.7, for $\sigma \geq 0$, we have $\Gamma^\sigma(K) \subset \tilde{\Gamma}^\sigma(K)$, with $\|f\|_{\tilde{\Gamma}^\sigma(K)} \lesssim \|f\|_{\Gamma^\sigma(K)}$. It is of interest to see how much information is added by introducing the cell graph structure in the definition of $\Gamma^\sigma(K)$ compared with $\tilde{\Gamma}^\sigma(K)$. We have a pair of theorems to answer this question.

Theorem 5.8. For $0 \leq \sigma < \frac{d_S}{2}$, we have $\Gamma^\sigma(K) = \tilde{\Gamma}^\sigma(K)$ with $\|\cdot\|_{\Gamma^\sigma(K)} \asymp \|\cdot\|_{\tilde{\Gamma}^\sigma(K)}$. In particular, $\Gamma^0(K) = L^2(K)$ with $\|\cdot\|_{\Gamma^0(K)} \asymp \|\cdot\|_{L^2(K)}$.

Theorem 5.9. For $\sigma > \frac{d_S}{2}$, we have $\Gamma^\sigma(K) = \tilde{\Gamma}^\sigma(K) \cap C(K)$, and $\Gamma^\sigma(K)$ is a closed subspace of $\tilde{\Gamma}^\sigma(K)$. In particular, for $\sigma \geq 1$, we have

$$\tilde{\Gamma}^\sigma(K) \cap C(K) = \text{constants}.$$

Theorem 5.8 gives an equivalent characterization of $L^2(K)$ (take $\sigma = 0$), which will play a key role in the later proof of $H^\sigma(K) = \Gamma^\sigma(K)$, $0 \leq \sigma < 1$, while Theorem 5.9 is an interesting observation from the full characterization of $H^\sigma(K)$. We prove Theorem 5.8 in this section, and postpone the proof of Theorem 5.9 until Section 7.

The proof of Theorem 5.8 relies on the following two lemmas.

Lemma 5.10. For $l \geq 1$, $\tilde{f}_l \in \tilde{J}_l$, we have $\|D_l \tilde{f}_l\|_{l^2(E_{c,l})} \asymp r^{-ld_H/2} \|\tilde{f}_l\|_{L^2(K)}$.

Proof. Obviously, we have $r^{-ld_H/2} \|\tilde{f}_l\|_{L^2(K)} \asymp \|\tilde{D}_l \tilde{f}_l\|_{l^2(\tilde{E}_{c,l})} \leq \|D_l \tilde{f}_l\|_{l^2(E_{c,l})}$. Conversely, the estimate $\|D_l \tilde{f}_l\|_{l^2(E_{c,l})} \lesssim r^{-ld_H/2} \|\tilde{f}_l\|_{L^2(K)}$ is also clear, since

$$\begin{aligned} \|D_l \tilde{f}_l\|_{l^2(E_{c,l})}^2 &= \sum_{\{w,w'\} \in E_{c,l}} (A_w(\tilde{f}_l) - A_{w'}(\tilde{f}_l))^2 \\ &\lesssim \sum_{\{w,w'\} \in E_{c,l}} (A_w^2(\tilde{f}_l) + A_{w'}^2(\tilde{f}_l)) \lesssim \sum_{w \in \Lambda_l} A_w^2(\tilde{f}_l) \asymp r^{-ld_H} \|\tilde{f}_l\|_{L^2(K)}^2, \end{aligned}$$

where we use the fact that the number of cells in Λ_l neighboring $F_w K$ is bounded by $\#V_0 \#C$ in the second inequality. \square

Lemma 5.11. (a). For any $\tilde{f}_l \in \tilde{J}_l$ and $l > m$, we have $D_m \tilde{f}_l = 0$.

(b). For any $\tilde{f}_l \in \tilde{J}_l$ and $l \leq m$, we have $\|D_m \tilde{f}_l\|_{l^2(E_{c,m})} \asymp \|D_l \tilde{f}_l\|_{l^2(E_{c,l})}$.

Proof. (a) is obvious, so we only need to prove (b). In fact, since

$$\begin{aligned} \|D_m \tilde{f}_l\|_{l^2(E_{c,m})}^2 &= \sum_{\{w,w'\} \in E_{c,m}} (A_w(\tilde{f}_l) - A_{w'}(\tilde{f}_l))^2 \\ &= \sum_{\{v,v'\} \in E_{c,l}} \sum_{\substack{\{w,w'\} \in E_{c,m}, \\ w \in W_v, w' \in W_{v'}}} (A_w(\tilde{f}_l) - A_{w'}(\tilde{f}_l))^2 \\ &= \sum_{\{v,v'\} \in E_{c,l}} \sum_{\substack{\{w,w'\} \in E_{c,m}, \\ w \in W_v, w' \in W_{v'}}} (A_v(\tilde{f}_l) - A_{v'}(\tilde{f}_l))^2, \end{aligned}$$

and $1 \leq \#\{\{w,w'\} \in E_{c,m} : w \in W_v, w' \in W_{v'}\} \leq \#V_0(\#C)^2$, the estimate follows. \square

Proof of Theorem 5.8. By Proposition 5.7, we already have $\Gamma^\sigma(K) \subset \tilde{\Gamma}^\sigma(K)$, with $\|f\|_{\tilde{\Gamma}^\sigma(K)} \lesssim \|f\|_{\Gamma^\sigma(K)}$. It remains to show the other direction.

For $f = C + \sum_{l=1}^{\infty} \tilde{f}_l$ in $\tilde{\Gamma}^\sigma(K)$, using Lemma 5.11, we get the estimate that

$$\begin{aligned} \left(\sum_{m=1}^{\infty} \lambda^{2m} \|D_m f\|_{l^2(E_{c,m})}^2 \right)^{1/2} &= \|\lambda^m \|D_m f\|_{l^2(E_{c,m})}\|_{l^2} = \|\lambda^m \sum_{l=1}^m D_m \tilde{f}_l\|_{l^2(E_{c,m})}\|_{l^2} \\ &\leq \|\lambda^m \sum_{l=1}^m \|D_m \tilde{f}_l\|_{l^2(E_{c,m})}\|_{l^2} \lesssim \|\lambda^m \sum_{l=1}^m \|D_l \tilde{f}_l\|_{l^2(E_{c,l})}\|_{l^2} \\ &= \left\| \sum_{l=0}^{m-1} \lambda^l \cdot \lambda^{m-l} \|D_{m-l} \tilde{f}_{m-l}\|_{l^2(E_{c,m-l})} \right\|_{l^2} \\ &\leq \left(\sum_{l=0}^{\infty} \lambda^l \right) \cdot \|\lambda^m \|D_m \tilde{f}_m\|_{l^2(E_{c,m})}\|_{l^2}, \end{aligned}$$

where we use Minkowski inequality and the fact that $\lambda < 1$ in the last inequality. Then applying Lemma 5.10, we get $\|f\|_{\Gamma^\sigma(K)} \lesssim \|f\|_{\tilde{\Gamma}^\sigma(K)}$. \square

6. SMOOTHED HAAR FUNCTIONS

In this section, we would like to establish weighted expansions of functions in $\Gamma^\sigma(K)$, analogous to Definition 5.2 of $\tilde{\Gamma}^\sigma(K)$. Using this, we will show the Sobolev space $H^\sigma(K)$ is identical with $\Gamma^\sigma(K)$ when $0 \leq \sigma < 1$.

Theorem 6.1. *For $0 \leq \sigma < 1$, $H^\sigma(K) = \Gamma^\sigma(K)$ with $\|\cdot\|_{H^\sigma(K)} \asymp \|\cdot\|_{\Gamma^\sigma(K)}$.*

We will prove Theorem 6.1 in the following three subsections. As an application of Theorem 6.1 and the weighted expansions of functions in $\Gamma^\sigma(K)$, we will prove Theorem 4.9, i.e. $H^\sigma(K) = B_\sigma^{2,2}(K)$ for $0 < \sigma < 1$ in the last subsection. The following *smoothed Haar functions* will play a key role.

Definition 6.2. *For $m \geq 1$, for any $\tilde{f}_m \in \tilde{J}_m$, define $S_m \tilde{f}_m$ to be the unique function in $\text{dom}\mathcal{E}$ such that*

$$A_w(S_m \tilde{f}_m) = A_w(\tilde{f}_m), \quad \forall w \in \Lambda_m, \quad (6.1)$$

and

$$\mathcal{E}(S_m \tilde{f}_m) = \min\{\mathcal{E}(f) : f \in \text{dom}\mathcal{E}, A_w(f) = A_w(\tilde{f}_m), \forall w \in \Lambda_m\}, \quad (6.2)$$

call it a *m-smoothed Haar function*. Define the space of *m-smoothed Haar functions* by $J_m = S_m \tilde{J}_m$.

Noticing that $(\text{dom}\mathcal{E}/\text{constants}, \mathcal{E})$ is a Hilbert space, and for any $\tilde{f}_m \in \tilde{J}_m$, the collection $\{f \in \text{dom}\mathcal{E}, A_w(f) = A_w(\tilde{f}_m), \forall w \in \Lambda_m\}$ is a closed linear subspace of $\text{dom}\mathcal{E}$, the minimum in the right hand of (6.2) will be attained by some function $f \in \text{dom}\mathcal{E}$. In addition, by a standard variational argument, one can see that $S_m \tilde{f}_m$ is the unique function in $\{f \in \text{dom}\mathcal{E}, A_w(f) = A_w(\tilde{f}_m), \forall w \in \Lambda_m\}$ such that

$$\mathcal{E}(S_m \tilde{f}_m, g) = 0, \quad (6.3)$$

for any $g \in \text{dom}\mathcal{E}$ such that $A_w(g) = 0, \forall w \in \Lambda_m$. This implies that S_m is a linear map.

6.1. A decomposition of $\text{dom}\mathcal{E}$. In this subsection, we will explore some properties of smoothed Haar functions.

First, we have an easy observation of the orthogonality of smoothed Haar function spaces as following. We write $X \perp Y$ in $\text{dom}\mathcal{E}$ if $\mathcal{E}(f, g) = 0$ for any $f \in X$ and $g \in Y$.

Lemma 6.3. *For any $m \neq m' \geq 1$, we have $J_m \perp J_{m'}$ in $\text{dom}\mathcal{E}$.*

Proof. Without loss of generality, we assume $m < m'$. Then by (6.1) we always have $A_w(S_{m'} \tilde{f}_{m'}) = 0, \forall w \in \Lambda_m, \forall \tilde{f}_{m'} \in \tilde{J}_{m'}$. Thus, by (6.3),

$$\mathcal{E}(S_m \tilde{f}_m, S_{m'} \tilde{f}_{m'}) = 0, \forall \tilde{f}_m \in \tilde{J}_m, \tilde{f}_{m'} \in \tilde{J}_{m'}.$$

□

Next, we estimate the energies of smoothed Haar functions.

Lemma 6.4. *For $m \geq 1$ and $f \in \text{dom}\mathcal{E}$, we have $\|D_m f\|_{l^2(E_{c,m})}^2 \lesssim r^m \mathcal{E}(f)$. Moreover, if $f \in \text{constants} \oplus (\oplus_{l=1}^m J_l)$, then $\|D_m f\|_{l^2(E_{c,m})}^2 \asymp r^m \mathcal{E}(f)$.*

Proof. First, we show $\|D_m f\|_{l^2(E_{c,m})}^2 \lesssim r^m \mathcal{E}(f)$ for any $f \in \text{dom}\mathcal{E}$. For each $\{w, w'\} \in E_{c,m}$, let $p \in F_w K \cap F_{w'} K$, by Morrey-Sobolev's inequality, we have the estimate that

$$\begin{aligned} (A_w(f) - A_{w'}(f))^2 &\lesssim (f(p) - A_w(f))^2 + (f(p) - A_{w'}(f))^2 \\ &\lesssim r^m (\mathcal{E}_{F_w K}(f) + \mathcal{E}_{F_{w'} K}(f)). \end{aligned}$$

The desired estimate $\|D_m f\|_{l^2(E_{c,m})}^2 \lesssim r^m \mathcal{E}(f)$ follows by summing up the estimates over all edges in $E_{c,m}$.

Next, we show $r^m \mathcal{E}(f) \lesssim \|D_m f\|_{l^2(E_{c,m})}^2$ if $f \in \text{constants} \oplus (\oplus_{l=1}^m J_l)$. By (6.3), one can see that $\mathcal{E}(f, f') = 0$ for any $f' \in \text{dom}\mathcal{E}$ such that $A_w(f') = 0, \forall w \in \Lambda_m$. This implies

$$\mathcal{E}(f) = \inf\{\mathcal{E}(g) : g \in \text{dom}\mathcal{E}, A_w(g) = A_w(f), \forall w \in \Lambda_m\}.$$

So, we only need to construct a function $g \in \text{dom}\mathcal{E}$ with $A_w(g) = A_w(f), \forall w \in \Lambda_m$, such that $r^m \mathcal{E}(g) \lesssim \|D_m g\|_{l^2(E_{c,m})}^2 = \|D_m f\|_{l^2(E_{c,m})}^2$. This can be done as follows. First, define a piecewise harmonic function g_0 such that for each $x \in \bigcup_{w \in \Lambda_m} F_w V_0$,

$$g_0(x) = \frac{1}{\#\{w \in \Lambda_m : x \in F_w K\}} \sum_{w \in \Lambda_m, x \in F_w K} A_w(f),$$

and g_0 is harmonic in $F_w K$ for each $w \in \Lambda_m$. Next, choose $g' \in \text{dom}_0 \mathcal{E}$ such that $A_0(g') = 1$. Define

$$g = g_0 + \sum_{w \in \Lambda_m} (A_w(f) - A_w(g_0)) g' \circ F_w^{-1}.$$

For each $w \in \Lambda_m$, we have $A_w(g) = A_w(f)$ and

$$\begin{aligned} r_w \mathcal{E}_{F_w K}(g) &= \mathcal{E}(g \circ F_w) \lesssim \mathcal{E}(g_0 \circ F_w) + (A_w(f) - A_w(g_0))^2 \mathcal{E}(g') \\ &\lesssim \sum_{w': \{w, w'\} \in E_{c,m}} (A_w(f) - A_{w'}(f))^2. \end{aligned}$$

Summing up the inequalities over $w \in \Lambda_m$, we see that g satisfies the required estimate. \square

By using Lemma 6.4, one can see that constants and $J_m, m \geq 1$ expand $\text{dom}\mathcal{E}$. Since $J_m, m \geq 1$ are not pairwise orthogonal with respect to $L^2(K)$, we provide a short proof.

Corollary 6.5. *For any $f \in \text{dom}\mathcal{E}$, there is a unique expansion $f = C + \sum_{m=1}^{\infty} f_m$ converging with respect to the norm $\sqrt{\mathcal{E}(\cdot) + \|\cdot\|_{L^2}^2}$, where $C \in \text{constants}$ and $f_m \in J_m$. In addition,*

$$|C|^2 + \mathcal{E}(f) \asymp \|f\|_{L^2(K)}^2 + \sum_{m=1}^{\infty} r^{-m} \|D_m f_m\|_{l^2(E_{c,m})}^2.$$

Proof. For any $f \in \text{dom}\mathcal{E}$, let $C = A_0(f)$, we can inductively find a sequence $\{f_m\}_{m=1}^{\infty}$ such that $\forall m \geq 1$,

$$\begin{cases} f_m \in J_m, \\ A_w(C + \sum_{l=1}^m f_l) = A_w(f), \forall w \in \Lambda_m. \end{cases}$$

Then clearly we have $\mathcal{E}(\sum_{l=1}^m f_l) \leq \mathcal{E}(f), \forall m \geq 1$. As a result,

$$\left\| \sum_{l=1}^m f_l \right\|_{L^\infty(K)}^2 \lesssim \mathcal{E}\left(\sum_{l=1}^m f_l\right) \leq \mathcal{E}(f).$$

Thus for any $m \geq 1$ and $w \in \Lambda_m$, we have

$$A_w(C + \sum_{l=1}^{\infty} f_l) = \lim_{m' \rightarrow \infty} A_w(C + \sum_{l=1}^{m'} f_l) = A_w(C + \sum_{l=1}^m f_l) = A_w(f),$$

since $A_w(f_l) = 0, \forall l > m$. This shows that $f = C + \sum_{m=1}^{\infty} f_m$. The expansion is unique by Lemma 6.3. Finally, the energy estimate is a direct consequence of Lemma 6.3 and 6.4. \square

6.2. A decomposition of $\Gamma^\sigma(K)$ with $0 \leq \sigma < 1$. The benefit of the smoothed Haar functions comes from two aspects. First, they keep features from Haar functions. Second, they are “smooth”. Below we give a lemma which follows from what we have discussed in the last subsection.

Lemma 6.6. (a). For any $m' \geq m \geq 1$ and any $f \in \text{constants} \oplus (\oplus_{l=1}^m J_l)$, we have

$$\|D_{m'} f\|_{l^2(E_{c,m'})} \lesssim r^{\frac{m'-m}{2}} \|D_m f\|_{l^2(E_{c,m})}.$$

(b). For any $m' > m \geq 1$ and any $f \in J_{m'}$, we have $D_m f = 0$.

Proof. (a) is a simple consequence of Lemma 6.4. (b) comes naturally from Definition 6.2. \square

Theorem 6.7. Let $0 \leq \sigma < 1$ and $\lambda := \lambda(\sigma) = r^{(d_H - \sigma d_W)/2}$. We have

(a). For any $f \in \Gamma^\sigma(K)$, there is a unique sequence of functions $\{f_m\}_{m=1}^{\infty}$ with $f_m \in J_m$ and a constant C such that $f = C + \sum_{m=1}^{\infty} f_m$.

(b). For any series $f = C + \sum_{m=1}^{\infty} f_m$, $f \in \Gamma^\sigma(K)$ if and only if

$$\sum_{m=1}^{\infty} \lambda^{2m} \|D_m f_m\|_{l^2(E_{c,m})}^2 < \infty. \quad (6.4)$$

In addition,

$$\|f\|_{\Gamma^\sigma(K)} \asymp (|C|^2 + \sum_{m=1}^{\infty} \lambda^{2m} \|D_m f_m\|_{l^2(E_{c,m})}^2)^{1/2}.$$

Proof. Let $\{f_l\}_{l=1}^{\infty}$ be a sequence such that $f_l \in J_l, \forall l \geq 1$, $f_0 = C$ be a constant, and (6.4) holds. We first show that $\sum_{l=0}^n f_l$ is Cauchy in $L^2(K)$ using Theorem 5.8. In fact, for any $n' \geq n$, we see that

$$\left\| \sum_{l=n}^{n'} f_l \right\|_{L^2(K)} \lesssim \left(\sum_{m=1}^{\infty} \|r^{md_H/2} D_m \left(\sum_{l=n}^{n'} f_l \right)\|_{l^2(E_{c,m})}^2 \right)^{1/2},$$

and thus

$$\begin{aligned}
\left\| \sum_{l=n}^{n'} f_l \right\|_{L^2(K)} &\lesssim \left\| \lambda^m \sum_{l=n}^{n'} \|D_m f_l\|_{l^2(E_{c,m})} \right\|_{l^2} \\
&\lesssim \left\| \lambda^m \sum_{l=n}^m r^{(m-l)/2} \|D_l f_l\|_{l^2(E_{c,l})} \right\|_{l^2} \\
&= \left\| \lambda^m \sum_{l=0}^{m-n} r^{l/2} \|D_{m-l} f_{m-l}\|_{l^2(E_{c,m-l})} \right\|_{l^2} \\
&= \left\| \sum_{l=0}^{m-n} (r^{1/2} \lambda)^l \cdot \lambda^{m-l} \|D_{m-l} f_{m-l}\|_{l^2(E_{c,m-l})} \right\|_{l^2} \\
&\leq \left\| \chi_{m \geq n} \lambda^m \|D_m f_m\|_{l^2(E_{c,m})} \right\|_{l^2} \sum_{l=0}^{\infty} (r^{1/2} \lambda)^l \\
&\lesssim \left\| \chi_{m \geq n} \lambda^m \|D_m f_m\|_{l^2(E_{c,m})} \right\|_{l^2},
\end{aligned} \tag{6.5}$$

where we used Lemma 6.6 in the second inequality, Minkowski inequality in the last but one inequality, and the fact that $r^{1/2} \lambda < 1$. Hence $f = \sum_{l=0}^{\infty} f_l$ converges in $L^2(K)$. As a consequence, for any $m \geq 1$, $w \in \Lambda_m$, we get

$$A_w(f) = \lim_{m' \rightarrow \infty} A_w(C + \sum_{l=1}^{m'} f_l) = A_w(C + \sum_{l=1}^m f_l), \tag{6.6}$$

since A_w is a bounded functional on $L^2(K)$. Thus,

$$D_m(f) = \sum_{l=1}^m D_m f_l.$$

Using the above equality and by a similar process as estimate (6.5), we get the estimate that

$$\|f\|_{\Gamma^\sigma(K)} \lesssim (|C|^2 + \sum_{m=1}^{\infty} \lambda^{2m} \|D_m f_m\|_{l^2(E_{c,m})}^2)^{1/2}. \tag{6.7}$$

Next, let $f \in \Gamma^\sigma(K)$. Let $C = A_\emptyset(f)$, we can inductively find a sequence $\{f_m\}_{m=1}^{\infty}$ such that

$$\begin{cases} f_m \in J_m, & \forall m \geq 1, \\ A_w(C + \sum_{l=1}^m f_l) = A_w(f), & \forall w \in \Lambda_m, m \geq 0. \end{cases}$$

For simplicity, we write $g_m = C + \sum_{l=1}^m f_l \in \text{constants} \oplus (\oplus_{l=1}^m J_l)$. Then by Lemma 6.6 (a), for $m \geq 1$, we have

$$\|D_{m+1} g_m\|_{l^2(E_{c,m+1})} \lesssim \|D_m g_m\|_{l^2(E_{c,m})} = \|D_m f\|_{l^2(E_{c,m})}.$$

As a result,

$$\begin{aligned}
\|D_{m+1} f_{m+1}\|_{l^2(E_{c,m+1})} &\leq \|D_{m+1} g_{m+1}\|_{l^2(E_{c,m+1})} + \|D_{m+1} g_m\|_{l^2(E_{c,m+1})} \\
&\lesssim \|D_{m+1} f\|_{l^2(E_{c,m+1})} + \|D_m f\|_{l^2(E_{c,m})}.
\end{aligned}$$

In particular, $\|D_1 f_1\|_{l^2(E_{c,1})} = \|D_1 f\|_{l^2(E_{c,1})}$. Thus we get

$$\sum_{m=1}^{\infty} \lambda^{2m} \|D_m f_m\|_{l^2(E_{c,m})}^2 \lesssim \sum_{m=1}^{\infty} \lambda^{2m} \|D_m f\|_{l^2(E_{c,m})}^2. \quad (6.8)$$

By the first part of the discussion, we can see that $f = C + \sum_{m=1}^{\infty} f_m$, by using (6.6).

Thus we have proved the theorem. The estimates in (b) come from (6.7) and (6.8). \square

6.3. Proof of $H^\sigma(K) = \Gamma^\sigma(K)$ with $0 \leq \sigma < 1$. We have decomposed both $\text{dom}\mathcal{E}$ and $\Gamma^\sigma(K)$ with $0 \leq \sigma < 1$ into summations of smoothed Haar functions in Corollary 6.5 and Theorem 6.7 respectively.

Lemma 6.8. *Given any function $f = C + \sum_{m=1}^{\infty} \tilde{f}_m \in L^2(K)$ with $\tilde{f}_m \in \tilde{J}_m$, let Sf be*

$$Sf = C + \sum_{m=1}^{\infty} S_m \tilde{f}_m.$$

Then for $0 \leq \sigma \leq 1$, S is a linear homeomorphism from $\tilde{\Gamma}^\sigma(K)$ onto $H^\sigma(K)$.

Proof. The linearity of S follows from the linearity of S_m and the uniqueness of the series expansion. Recall the fact that $\|D_m S_m \tilde{f}_m\|_{l^2(E_{c,m})} = \|D_m \tilde{f}_m\|_{l^2(E_{c,m})} \asymp r^{-md_H/2} \|\tilde{f}_m\|_{L^2(K)}$ by Lemma 5.10, and recall Definition 5.3 of $\tilde{\Gamma}^\sigma(K)$. It follows from Theorem 6.7 and Corollary 6.5 that S is a homeomorphism from $\tilde{\Gamma}^\sigma(K)$ onto $\Gamma^\sigma(K)$ for any $0 \leq \sigma < 1$, and from $\tilde{\Gamma}^1(K)$ onto $\text{dom}\mathcal{E}$.

Notice that $H^0(K) = \Gamma^0(K)$ by Theorem 5.8, and $H^1(K) = \text{dom}\mathcal{E}$ by Corollary 3.3. So the lemma is true for $\sigma = 0$ and $\sigma = 1$. Since both $\tilde{\Gamma}^\sigma(K)$ and $H^\sigma(K)$ are stable under complex interpolation, i.e. $\tilde{\Gamma}^\sigma(K) = [\tilde{\Gamma}^0(K), \tilde{\Gamma}^1(K)]_\sigma$ and $H^\sigma(K) = [H^0(K), H^1(K)]_\sigma$, the lemma then follows. \square

Theorem 6.1 is an immediate consequence of Lemma 6.8 and Theorem 6.7.

6.4. Proof of $H^\sigma(K) = B_\sigma^{2,2}(K)$ with $0 < \sigma < 1$. As an application of Theorem 6.1 and 6.7, we provide a pure analytic proof of Theorem 4.9. First, we list some easy estimates. Recall Definition 4.8 for the definition of $B_\sigma^{2,2}(K)$, and still write $\lambda = \lambda(\sigma) = r^{(d_H - \sigma d_W)/2}$ for short.

Lemma 6.9. *For $f \in L^2(K)$, $m \geq 0$, write*

$$I_m(f) = r^{-md_H} \left(\int_K \int_{B(x, r^m)} |f(x) - f(y)|^2 d\mu(y) d\mu(x) \right)^{1/2}.$$

Then for $\sigma > 0$, we have $[f]_{B_\sigma^{2,2}(K)} \asymp \|\lambda^m I_m(f)\|_{l^2}$.

Proof. It is not hard to see that

$$[f]_{B_\sigma^{2,2}(K)} \asymp \left(\sum_{m=1}^{\infty} r^{-md_H - m\sigma d_W} \int_K \int_{B(x, r^m)} |f(x) - f(y)|^2 d\mu(y) d\mu(x) \right)^{1/2}.$$

The lemma follows immediately since $(r^{-md_H} \lambda^m)^2 = r^{-md_H - m\sigma d_W}$. \square

Lemma 6.10. *$I_m(f_l) \lesssim r^{\frac{l-m}{2}d_H} \|D_l f_l\|_{l^2(E_{c,l})}$ for $f_l \in J_l$ and $0 \leq m < l$.*

Proof. We prove the lemma through 3 steps.

First, for $w \in \Lambda_l$, writing $M_w(f_l) = \max_{x \in F_w K} |f_l(x)|$, we will prove that

$$\left(\sum_{w \in \Lambda_l} M_w^2(f_l) \right)^{1/2} \lesssim \|D_l f_l\|_{l^2(E_{c,l})}. \quad (6.9)$$

In fact, this follows from energy estimates. Let $x \in F_w K$ such that $|f_l(x)| = M_w(f_l)$, and let $y \in F_w K$ such that $f_l(y) = A_w(f_l)$. Then

$$\mathcal{E}_{F_w(K)}(f_l) \geq R^{-1}(x, y)(f_l(x) - f_l(y))^2 \gtrsim r^{-l}(f_l(x) - f_l(y))^2.$$

Thus by Lemma 6.4, we have

$$r^{-l} \sum_{w \in \Lambda_l} (|A_w(f_l)| - M_w(f_l))^2 \lesssim \mathcal{E}(f_l) \asymp r^{-l} \|D_l f_l\|_{l^2(E_{c,l})}^2.$$

This gives (6.9).

Next, we claim that

$$\|f_l\|_{L^2(K)} \asymp r^{ld_H/2} \|D_l f_l\|_{l^2(E_{c,l})}. \quad (6.10)$$

This follows from (6.9) and the observation that

$$r^{ld_H} \|D_l f_l\|_{l^2(E_{c,l})}^2 \asymp r^{ld_H} \sum_{w \in \Lambda_l} |A_w(f_l)|^2 \lesssim \|f_l\|_{L^2(K)}^2 \lesssim r^{ld_H} \sum_{w \in \Lambda_l} |M_w(f_l)|^2.$$

Finally, for $0 \leq m < l$, we have

$$\begin{aligned} I_m^2(f_l) &= r^{-2md_H} \int_K \int_{B(x, r^m)} |f_l(x) - f_l(y)|^2 d\mu(y) d\mu(x) \\ &\leq 2 \cdot r^{-2md_H} \int_K \int_{B(x, r^m)} (|f_l(x)|^2 + |f_l(y)|^2) d\mu(y) d\mu(x) \\ &\lesssim r^{-md_H} \|f_l\|_{L^2(K)}^2 \lesssim r^{(l-m)d_H} \|D_l f_l\|_{l^2(E_{c,l})}^2, \end{aligned}$$

where we use (6.10) in the last inequality. \square

Lemma 6.11. For $1 \leq m \leq m'$, and $f \in \text{constants} \oplus_{l=1}^{m'} J_l$, we have

$$I_m(f) \lesssim r^{\frac{m-m'}{2}} \|D_{m'} f\|_{l^2(E_{c,m'})}.$$

Proof. First, we claim that for $f \in \text{dom} \mathcal{E}$, we always have

$$I_m(f) \lesssim r^{m/2} \mathcal{E}^{1/2}(f). \quad (6.11)$$

In fact, we only need to notice that for any $w \in \Lambda_m$, we have

$$r^{-2md_H} \int_{F_w K} \int_{B(x, r^m)} |f(x) - f(y)|^2 d\mu(y) d\mu(x) \lesssim r^m \mathcal{E}_A(f),$$

where $A = \bigcup \{F_{w'} K : w' \in \Lambda_m, \exists x \in F_w K \text{ so that } F_{w'} K \cap B(x, r^m) \neq \emptyset\}$, as $|f(x) - f(y)|^2 \lesssim r^m \mathcal{E}_A(f)$. Then (6.11) follows by summing the above estimate over all $w \in \Lambda_m$, noticing that $\#A$ is bounded from above by a constant independent of m due to Proposition 2.3.

The lemma then follows as a consequence of Lemma 6.4 and (6.11). \square

Remark. In [13], Gu and Lau studied another class of Besov spaces, $B_\sigma^{2,\infty}(K)$, which includes $\text{dom} \mathcal{E}$ as a critical case. They proved that for $f \in \text{dom} \mathcal{E}$, $\sup_{m \geq 0} r^{-m} I_m^2(f) \asymp \mathcal{E}(f)$.

Proof of Theorem 4.9. By Theorem 6.1, it suffices to prove $\Gamma^\sigma(K) = B_\sigma^{2,2}(K)$.

First, let's prove that $B_\sigma^{2,2}(K) \subset \Gamma^\sigma(K)$. Without loss of generality, assume that $\text{diam}(K) = \max\{R(x, y) : x, y \in K\} = 1$. Then for $f \in B_\sigma^{2,2}(K)$, for $m \geq 1$ and $\{w, w'\} \in E_{c,m}$, we have

$$\begin{aligned} |A_w(f) - A_{w'}(f)|^2 &\leq \frac{1}{\mu(F_w K)\mu(F_{w'} K)} \int_{F_w K} \int_{F_{w'} K} |f(x) - f(y)|^2 d\mu(x)d\mu(y) \\ &\lesssim r^{-2md_H} \int_{F_w K} \int_{B(x, 2r^m)} |f(x) - f(y)|^2 d\mu(y)d\mu(x), \end{aligned}$$

by Jensen's inequality. Summing the above estimate over all edges in $E_{c,m}$, we then get that

$$\|D_m f\|_{l^2(E_{c,m})} \lesssim I_{m+\lceil \log 2 / \log r \rceil}(f).$$

This gives that $f \in \Gamma^\sigma(K)$ and $\|f\|_{\Gamma^\sigma(K)} \lesssim \|f\|_{B_\sigma^{2,2}(K)}$ by Lemma 6.9.

Next, for the other direction, recall that by Theorem 6.7, for any $f \in \Gamma^\sigma(K)$, there is a unique expansion that $f = C + \sum_{l=1}^{\infty} f_l$ with $f_l \in J_l$. For $m \geq 1$, we write $g_m = C + \sum_{l=1}^m f_l$. Then we have the estimate that

$$\begin{aligned} \|\lambda^m I_m(f)\|_{l^2} &\lesssim \|\lambda^m I_m(g_m) + \lambda^m \sum_{l=m+1}^{\infty} I_m(f_l)\|_{l^2} \\ &\leq \|\lambda^m I_m(g_m)\|_{l^2} + \|\lambda^m \sum_{l=m+1}^{\infty} I_m(f_l)\|_{l^2} \\ &\lesssim \|\lambda^m \|D_m g_m\|_{l(E_{c,m})}\|_{l^2} + \|\lambda^m \sum_{l=m+1}^{\infty} r^{\frac{l-m}{2}d_H} \|D_l f_l\|_{l^2(E_{c,l})}\|_{l^2} \\ &= \|\lambda^m \|D_m f\|_{l(E_{c,m})}\|_{l^2} + \|\lambda^m \sum_{l=1}^{\infty} r^{ld_H/2} \|D_{m+l} f_{m+l}\|_{l^2(E_{c,m+l})}\|_{l^2} \\ &\leq \|\lambda^m \|D_m f\|_{l(E_{c,m})}\|_{l^2} + \sum_{l=1}^{\infty} (r^{-d_H/2} \lambda)^{-l} \cdot \|\lambda^m \|D_m f_m\|_{l^2(E_{c,m})}\|_{l^2} \\ &\lesssim \|f\|_{\Gamma^\sigma(K)}, \end{aligned}$$

where we use Lemma 6.10 and 6.11 in the third inequality, use Minkowski inequality in the fourth inequality, and use Theorem 6.7 in the last inequality, noticing that $r^{-d_H/2} \lambda > 1$. This finishes the proof. \square

Remark. In fact, in the above proof, we have proved that $B_\sigma^{2,2}(K) \subset \Gamma^\sigma(K)$ for all $\sigma > 0$. Combining this with Theorem 5.9, we immediately get that $B_\sigma^{2,2}(K) = \text{constants}$ whenever $\sigma \geq 1$.

7. ATOMIC DECOMPOSITIONS: LOWER ORDERS

In Section 5 and 6, we have developed all the tools for the proof of the atomic decomposition theorem Theorem 4.1 for $0 \leq \sigma < 1$. Now we will come to the proof of this case in this section. In the next section, we then sketch the proof for $1 \leq \sigma < 2$. The general $\sigma \geq 0$ case then immediately follows with formula (3.7) by combining the results for $0 \leq \sigma < 1$ and $1 \leq \sigma < 2$ together.

As applications of the case $0 \leq \sigma < 1$, additionally in this section, we will prove Theorem 4.7 (a), i.e. for $\frac{d_S}{2} < \sigma < 1$, $H^\sigma(K)$ is identical with another class of Besov type spaces $\Lambda_\sigma^{2,2}(K)$. Also, we will prove Theorem 5.9.

As in previous sections, we still abbreviate that $\lambda = \lambda(\sigma) = r^{(d_H - \sigma d_W)/2}$ for $\sigma \geq 0$.

7.1. Proof of Theorem 4.1 (a) for $k = 0$, so $0 \leq \sigma < \frac{d_S}{2}$. When $\sigma < \frac{d_S}{2}$, we already have $H^\sigma(K) = \tilde{\Gamma}^\sigma(K)$ by Theorem 5.8 and 6.1. This immediately implies Theorem 4.1 for the case $0 \leq \sigma < \frac{d_S}{2}$.

Proof of Theorem 4.1 (a) for $0 \leq \sigma < \frac{d_S}{2}$. For each $m \geq 1$, write

$$\Lambda_m^- = \{w \in W_* : \exists w' \in \Lambda_{m-1}, w'' \in \Lambda_m, \text{ such that } F_{w''}K \subsetneq F_w K \subset F_{w'}K\}, \quad (7.1)$$

and we can easily check that $W_* = \bigcup_{m=1}^\infty \Lambda_m^-$.

Define

$$\tilde{f}_m = \sum_{w \in \Lambda_m^-} \sum_{i=1}^N a_{wi} \chi_{F_{wi}K},$$

Then, it is direct to check that $\tilde{f}_m \in \tilde{J}_m, \forall m \geq 1$. In addition,

$$r^{-md_H} \lambda^{2m} \|\tilde{f}_m\|_{L^2(K)}^2 = r^{-m\sigma d_W} \sum_{w \in \Lambda_m^-} \sum_{i=1}^N r_{wi}^{d_H} |a_{wi}|^2 \asymp \sum_{w \in \Lambda_m^-} r_w^{d_H - \sigma d_W} \sum_{i=1}^N |a_{wi}|^2.$$

The result follows from the above estimate immediately, noticing that $\|f\|_{H^\sigma(K)} \asymp (|C|^2 + \sum_{m=1}^\infty r^{-md_H} \lambda^{2m} \|\tilde{f}_m\|_{L^2(K)}^2)^{1/2}$. \square

7.2. Atomic decomposition: $\frac{d_S}{2} < \sigma < 1$. In this part, we will prove Theorem 4.1 for $\frac{d_S}{2} < \sigma < 1$. Recall the definition of tent functions ψ_x in Section 4.1.

Definition 7.1. For $m \geq 1$, denote T_m the linear subspace spanned by $\{\psi_{F_w x} : w \in \Lambda_m^-, x \in V_1 \setminus V_0\}$, where Λ_m^- is defined in (7.1).

When $\sigma > \frac{d_S}{2}$, it is well-known that $H^\sigma(K) \subset C(K)$. In fact, this is an immediate consequence of Theorem 6.1 and 6.7. Obviously, each function $f \in C(K)$ can be written uniquely as a uniformly convergent series of tent functions

$$f = \varphi_0 + \sum_{x \in V_* \setminus V_0} c_x \psi_x, \quad (7.2)$$

with $\varphi_0 \in \mathcal{H}_0$ and $c_x \in \mathbb{R}$. By the definition of T_m , immediately we see that f admits a unique expansion of the form

$$f = \sum_{m=0}^\infty \varphi_m, \quad (7.3)$$

with $\varphi_0 \in \mathcal{H}_0$ and $\varphi_m \in T_m, \forall m \geq 1$.

The following lemma follows by a similar argument as the proof in Section 7.1.

Lemma 7.2. *Let $f \in C(K)$, and take the expansions in (7.2) and (7.3). Then, for $\sigma > \frac{d_S}{2}$, we have $\sum_{w \in W_*} r_w^{d_H - \sigma d_W} \sum_{x \in V_1 \setminus V_0} |c_{F_w x}|^2 < \infty$ if and only if $\sum_{m=0}^{\infty} r^{-m\sigma d_W} \|\varphi_m\|_{L^2(K)}^2 < \infty$. In addition,*

$$\|\varphi_0\|_{L^2(K)}^2 + \sum_{w \in W_*} r_w^{d_H - \sigma d_W} \sum_{x \in V_1 \setminus V_0} |c_{F_w x}|^2 \asymp \sum_{m=0}^{\infty} r^{-m\sigma d_W} \|\varphi_m\|_{L^2(K)}^2.$$

One can compare Lemma 7.2 with Theorem 6.7. In fact, we will see that there is mutual control of norms of functions in J_m and T_m . The same idea will be used in the proof for higher orders in Section 8.

Recall ∇_m defined in Definition 4.2. It is direct to check that for $m \geq 1$ and $\varphi_m \in T_m$, we always have

$$r^{-md_H/2} \|\varphi_m\|_{L^2(K)} \asymp \|\nabla_m \varphi_m\|_{l^2(E_{v,m})} \asymp r^{m/2} \mathcal{E}^{1/2}(\varphi_m). \quad (7.4)$$

The following lemmas provide a mutual control of functions in J_m and T_m .

Lemma 7.3. *Let $l \geq 1$, $f_l = \sum_{m=0}^{\infty} \varphi_{m,l} \in J_l$ with $\varphi_{0,l} \in \mathcal{H}_0$ and $\varphi_{m,l} \in T_m$.*

- (a). *If $m \leq l$, then $\|\nabla_m \varphi_{m,l}\|_{l^2(E_{v,m})} \lesssim \|D_l f_l\|_{l^2(E_{c,l})}$.*
- (b). *If $m > l$, then $\|\nabla_m \varphi_{m,l}\|_{l^2(E_{v,m})} \lesssim r^{(m-l)/2} \|D_l f_l\|_{l^2(E_{c,l})}$.*

Proof. Since $\varphi_{m,l} = \sum_{m'=0}^m \varphi_{m',l} - \sum_{m'=0}^{m-1} \varphi_{m',l}$, we have

$$\|\nabla_m \varphi_{m,l}\|_{l^2(E_{v,m})} \leq \left\| \nabla_m \left(\sum_{m'=0}^m \varphi_{m',l} \right) \right\|_{l^2(E_{v,m})} + \left\| \nabla_m \left(\sum_{m'=0}^{m-1} \varphi_{m',l} \right) \right\|_{l^2(E_{v,m})},$$

and thus

$$\|\nabla_m \varphi_{m,l}\|_{l^2(E_{v,m})} \lesssim \|\nabla_m f_l\|_{l^2(E_{v,m})} + \|\nabla_{m-1} f_l\|_{l^2(E_{v,m-1})}.$$

In case (a) we use the obvious pointwise bound and the estimate (6.9) to write

$$\|\nabla_m f_l\|_{l^2(E_{v,m})}^2 \lesssim \sum_{w \in \Lambda_m} M_w^2(f_l) \lesssim \|D_l f_l\|_{l^2(E_{c,l})}^2.$$

In case (b) we instead use Lemma 6.4 to get

$$\|\nabla_m \varphi_{m,l}\|_{l^2(E_{v,m})} \lesssim r^{m/2} \mathcal{E}^{1/2}(f_l) \lesssim r^{(m-l)/2} \|D_l f_l\|_{l^2(E_{c,l})}.$$

□

Lemma 7.4. *Let $l \geq 0$, $\varphi_l = C_l + \sum_{m=1}^{\infty} f_{m,l} \in T_l$ (or \mathcal{H}_0) with $C_l \in \mathbb{R}$ and $f_{m,l} \in J_m$.*

- (a). *If $m \leq l$, then $\|D_m f_{m,l}\|_{l^2(E_{c,m})} \lesssim r^{(l-m)d_H/2} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})}$.*
- (b). *If $m > l$, then $\|D_m f_{m,l}\|_{l^2(E_{c,m})} \lesssim r^{(m-l)/2} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})}$.*

Proof. (a). By Theorem 5.8 (taking $\sigma = 0$), and formula (7.4), we have

$$\|D_m f_{m,l}\|_{l^2(E_{c,m})} \lesssim r^{-md_H/2} \|\varphi_l\|_{L^2(K)} \lesssim r^{(l-m)d_H/2} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})}.$$

(b). By Lemma 6.3, Lemma 6.4 and formula (7.4), we have

$$\|D_m f_{m,l}\|_{l^2(E_{c,m})} \lesssim r^{m/2} \mathcal{E}^{1/2}(f_{m,l}) \leq r^{m/2} \mathcal{E}^{1/2}(\varphi_l) \lesssim r^{(m-l)/2} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})}.$$

□

Proof of Theorem 4.1 (b) for $\frac{d_S}{2} < \sigma < 1$. Let $f \in C(K)$. Write f with the expansions that $f = \sum_{l=0}^{\infty} \varphi_l$ and $f = C + \sum_{l=1}^{\infty} f_l$ with $\varphi_0 \in \mathcal{H}_0$, $C \in \mathbb{R}$, $\varphi_l \in T_l$ and $f_l \in J_l$ for $l \geq 1$. By Lemma 7.2, Theorem 6.1, Theorem 6.7, and using formula (7.4), it suffices to prove that $\sum_{l=1}^{\infty} \lambda^{2l} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})}^2 < \infty$ if and only if $\sum_{l=1}^{\infty} \lambda^{2l} \|D_l f_l\|_{l^2(E_{c,l})}^2 < \infty$, and

$$\|\varphi_0\|_{L^2(K)} + \sum_{l=1}^{\infty} \lambda^{2l} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})}^2 \asymp |C|^2 + \sum_{l=1}^{\infty} \lambda^{2l} \|D_l f_l\|_{l^2(E_{c,l})}^2.$$

It suffices to consider both the function series converging in $L^2(K)$ in the rest of this proof.

Assume that $\sum_{l=1}^{\infty} \lambda^{2l} \|\nabla_l \varphi_l\|_{l^2(E_{c,l})}^2 < \infty$. For each $l \geq 0$, we expand $\varphi_l = C_l + \sum_{m=1}^{\infty} f_{m,l}$ with $C_l \in \mathbb{R}$ and $f_{m,l} \in J_m$. By Lemma 7.4, one has

$$\sum_{l=0}^{\infty} \|D_m f_{m,l}\|_{l^2(E_{c,m})} \lesssim \sum_{l=0}^{m-1} r^{(m-l)/2} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})} + \sum_{l=m}^{\infty} r^{(l-m)d_H/2} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})} < \infty.$$

So $\sum_{l=0}^{\infty} f_{m,l}$ converges in J_m , hence in $L^2(K)$, and we write $g_m = \sum_{l=0}^{\infty} f_{m,l}$ for convenience. Next, let $C' = \sum_{l=0}^{\infty} C_l$. We observe that

$$A_w(C' + \sum_{m'=1}^m g_{m'}) = \lim_{l \rightarrow \infty} A_w\left(\sum_{l'=0}^l C_{l'} + \sum_{m'=1}^m \sum_{l'=0}^l f_{m',l'}\right) = \lim_{l \rightarrow \infty} A_w\left(\sum_{l'=0}^l \varphi_{l'}\right) = A_w(f),$$

for any $m \geq 0, w \in W_m$. By the uniqueness of the expansion proven in Theorem 6.7 and the characterization of the expansion (6.6), this implies that $f_m = g_m = \sum_{l=0}^{\infty} f_{m,l}, \forall m \geq 1$. Thus

$$\begin{aligned} \|\lambda^m \|D_m f_m\|_{l^2(E_{c,m})}\|_{l^2} &\leq \left\| \lambda^m \sum_{l=0}^{\infty} \|D_m f_{m,l}\|_{l^2(E_{c,m})} \right\|_{l^2} \\ &\leq \left\| \lambda^m \sum_{l=0}^{m-1} \|D_m f_{m,l}\|_{l^2(E_{c,m})} \right\|_{l^2} + \left\| \lambda^m \sum_{l=m}^{\infty} \|D_m f_{m,l}\|_{l^2(E_{c,m})} \right\|_{l^2}. \end{aligned}$$

By Lemma 7.4 (b), using Minkowski inequality, noticing that $\lambda r^{1/2} < 1$, we have

$$\begin{aligned} \left\| \lambda^m \sum_{l=0}^{m-1} \|D_m f_{m,l}\|_{l^2(E_{c,m})} \right\|_{l^2} &\lesssim \left\| \lambda^m \sum_{l=0}^{m-1} r^{(m-l)/2} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})} \right\|_{l^2} \\ &= \left\| \sum_{l=1}^m r^{l/2} \lambda^l \lambda^{m-l} \|\nabla_{m-l} \varphi_{m-l}\|_{l^2(E_{v,m-l})} \right\|_{l^2} \\ &\lesssim \left\| \lambda^m \|\nabla_m \varphi_m\|_{l^2(E_{v,m})} \right\|_{l^2}. \end{aligned}$$

On the other hand, by Lemma 7.4 (a), using a similar argument, we have

$$\begin{aligned} \left\| \lambda^m \sum_{l=m}^{\infty} \|D_m f_{m,l}\|_{l^2(E_{c,m})} \right\|_{l^2} &\lesssim \left\| \lambda^m \sum_{l=m}^{\infty} r^{(l-m)d_H/2} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})} \right\|_{l^2} \\ &= \left\| \sum_{l=0}^{\infty} \lambda^{-l} r^{ld_H/2} \lambda^{l+m} \|\nabla_{l+m} \varphi_{l+m}\|_{l^2(E_{v,l+m})} \right\|_{l^2} \\ &\lesssim \left\| \lambda^m \|\nabla_m \varphi_m\|_{l^2(E_{v,m})} \right\|_{l^2}. \end{aligned}$$

Combining the above estimates, together with the observation that $|C| \leq \sum_{m=0}^{\infty} \|\varphi_m\|_{L^\infty(K)}$ and $\|\nabla_0 \varphi_0\|_{l^2(E_{v,0})} \lesssim \|\varphi_0\|_{L^2(K)}$, we get that

$$|C|^2 + \sum_{m=1}^{\infty} \lambda^{2m} \|D_m f_m\|_{l^2(E_{c,m})}^2 \lesssim \|\varphi_0\|_{L^2(K)}^2 + \sum_{m=1}^{\infty} \lambda^{2m} \|\nabla_m \varphi_m\|_{l^2(E_{v,m})}^2.$$

For the other direction estimate, similarly as above, we will assume that $\sum_{l=1}^{\infty} \lambda^{2l} \|D_l f_l\|_{l^2(E_{c,l})}^2 < \infty$, and expand $f_l = \sum_{m=0}^{\infty} \varphi_{m,l}$ with $\varphi_{0,l} \in \mathcal{H}_0$ and $\varphi_{m,l} \in T_m$. Then the estimate follows by a similar argument as above by using Lemma 7.3 instead. \square

7.3. Proof of $H^\sigma(K) = \Lambda_\sigma^{2,2}(K)$ with $\frac{d_S}{2} < \sigma < 1$. Now we come to the proof of Theorem 4.7 (a). Recall Definition 4.3 for the Besov type spaces $\Lambda_\sigma^{2,2}(K)$.

Notice that in the last subsection we have shown that $f = \sum_{m=0}^{\infty} \varphi_m \in H^\sigma(K)$ if and only if $\sum_{m=1}^{\infty} \lambda^{2m} \|\nabla_m \varphi_m\|_{l^2(E_{v,m})}^2 < \infty$, and $\|f\|_{H^\sigma(K)} \asymp (\|\varphi_0\|_{L^2(K)} + \sum_{m=1}^{\infty} \lambda^{2m} \|\nabla_m \varphi_m\|_{l^2(E_{v,m})}^2)^{1/2}$. Thus to prove Theorem 4.7 (a), it suffices to prove a same result with $\Lambda_\sigma^{2,2}(K)$ instead of $H^\sigma(K)$. The proof is essential the same as that of Theorem 6.7, which relies on Lemma 6.6. Below we provide a lemma analogous to Lemma 6.6, and omit the proof of Theorem 4.7 (a).

Lemma 7.5. (a). For any $m' \geq m \geq 1$ and any $f \in \mathcal{H}_0 \oplus (\oplus_{l=1}^m T_l)$, we have

$$\|\nabla_{m'} f\|_{l^2(E_{v,m'})} \lesssim r^{\frac{m'-m}{2}} \|\nabla_m f\|_{l^2(E_{v,m})}.$$

(b). For any $m' > m \geq 1$ and any $f \in T_{m'}$, we have $\nabla_m f = 0$.

Proof. (a). We have the estimate that

$$\mathcal{E}^{1/2}(f) \asymp r^{-m'/2} \|\nabla_{m'} f\|_{l^2(E_{v,m'})},$$

since f is harmonic in $F_w K$ for any $w \in \Lambda_m$. Thus,

$$\|\nabla_{m'} f\|_{l^2(E_{v,m'})} \asymp r^{m'/2} \mathcal{E}^{1/2}(f) \asymp r^{(m'-m)/2} \|\nabla_m f\|_{l^2(E_{v,m})}.$$

(b) is trivially true since $f|_{F_w V_0} = 0$ for any $w \in \Lambda_{m'-1}$. \square

7.4. Proof of $\Gamma^\sigma(K) = \tilde{\Gamma}^\sigma(K) \cap C(K)$ for $\sigma > \frac{d_S}{2}$. In this part, we prove Theorem 5.9, which illustrates the relation between $\Gamma^\sigma(K)$ and $\tilde{\Gamma}^\sigma(K)$ when $\sigma > \frac{d_S}{2}$. Since $H^\sigma(K) = \Gamma^\sigma(K)$ when $\sigma < 1$, this gives another characterization of $H^\sigma(K)$.

First, we introduce the following notations.

For each $\omega \in \pi^{-1}(V_*)$, define

$$\{w(k, \omega)\}_{k=0}^{\infty} = \{[\omega]_l : l \geq 0, [\omega]_l \in \bigcup_{m=0}^{\infty} \Lambda_m \text{ and } \pi(\omega) \in F_{[\omega]_l} V_0\},$$

with order

$$|w(0, \omega)| < |w(1, \omega)| < |w(2, \omega)| < \dots.$$

We then easily have the following properties.

1. If $\pi(\omega) \in (\bigcup_{w \in \Lambda_m} F_w V_0) \setminus (\bigcup_{w \in \Lambda_{m-1}} F_w V_0)$, then $w(k, \omega) \in \Lambda_{m+k}$;
2. Let $w \in \Lambda_m$ and $\omega \in \pi^{-1}(V_*)$, we have $w \in \{w(k, \omega)\}_{k=0}^{\infty}$ if and only if $[\omega]_{|w|} = w$ and $\sigma^{|w|}(\omega) \in \mathcal{P}$;
3. $\bigcup_{\omega \in \pi^{-1}(V_*)} \{w(k, \omega)\}_{k=0}^{\infty} = \bigcup_{m=0}^{\infty} \Lambda_m$.

We have the following estimates.

Lemma 7.6. *Let $\sigma \geq 0$. For $f = C + \sum_{m=1}^{\infty} \tilde{f}_m \in \tilde{\Gamma}^{\sigma}(K)$ with $C \in \mathbb{R}$ and $\tilde{f}_m \in \tilde{J}_m$, we have*

$$\sum_{\omega \in \pi^{-1}(V_*)} \sum_{k=0}^{\infty} r_{w(k,\omega)}^{d_H - \sigma d_W} (A_{w(k+1,\omega)}(f) - A_{w(k,\omega)}(f))^2 \lesssim \sum_{m=1}^{\infty} r^{-md_H} \lambda^{2m} \|\tilde{f}_m\|_{L^2(K)}^2.$$

Proof. Recall $W_w = \{w' \in W_* : F_{w'}K \subset F_w K\}$ in Definition 5.6. It is direct to check that

$$\|\tilde{f}_m\|_{L^2(K)}^2 = \sum_{w \in \Lambda_{m-1}} \sum_{w' \in W_w \cap \Lambda_m} r_{w'}^{d_H} (A_{w'}(f) - A_w(f))^2,$$

and thus

$$\begin{aligned} \sum_{m=1}^{\infty} r^{-md_H} \lambda^{2m} \|\tilde{f}_m\|_{L^2(K)}^2 &\asymp \sum_{m=1}^{\infty} \lambda^{2m} \sum_{w \in \Lambda_{m-1}} \sum_{w' \in W_w \cap \Lambda_m} (A_{w'}(f) - A_w(f))^2 \\ &\gtrsim \sum_{\omega \in \pi^{-1}(V_*)} \sum_{k=0}^{\infty} r_{w(k,\omega)}^{d_H - \sigma d_W} (A_{w(k+1,\omega)}(f) - A_{w(k,\omega)}(f))^2, \end{aligned}$$

since each $(A_{w'}(f) - A_w(f))^2$ occurs at most $\#\mathcal{P}$ times in the last term of the above estimate by Property 2. \square

Lemma 7.7. *Let $\sigma > \frac{d_S}{2}$. For $f \in \tilde{\Gamma}^{\sigma}(K) \cap C(K)$, we have*

$$\sum_{\omega \in \pi^{-1}(V_*)} \sum_{k=0}^{\infty} r_{w(k,\omega)}^{d_H - \sigma d_W} (A_{w(k,\omega)}(f) - f(\pi(\omega)))^2 \lesssim \|f\|_{\tilde{\Gamma}^{\sigma}(K)}^2.$$

Proof. Fix $\omega \in \pi^{-1}(V_*)$. By Property 1, for any $k \geq 0$, we have $w(k,\omega) \in \Lambda_{k+m}$ for some $m \geq 0$. Noticing that $f(\pi(\omega)) = \lim_{k \rightarrow \infty} A_{w(k,\omega)}(f)$, we have

$$\begin{aligned} \|r_{w(k,\omega)}^{(d_H - \sigma d_W)/2} (A_{w(k,\omega)}(f) - f(\pi(\omega)))\|_{l^2} &\asymp \lambda^m \|\lambda^k (A_{w(k,\omega)}(f) - f(\pi(\omega)))\|_{l^2} \\ &= \lambda^m \left\| \sum_{l=0}^{\infty} \lambda^{-l} \lambda^{k+l} (A_{w(k+l,\omega)}(f) - A_{w(k+l+1,\omega)}(f)) \right\|_{l^2} \\ &\lesssim \lambda^m \|\lambda^k (A_{w(k,\omega)}(f) - A_{w(k+1,\omega)}(f))\|_{l^2} \\ &\asymp \|r_{w(k,\omega)}^{(d_H - \sigma d_W)/2} (A_{w(k+1,\omega)}(f) - A_{w(k,\omega)}(f))\|_{l^2}, \end{aligned}$$

by using Minkowski inequality and the fact that $\lambda > 1$. Thus,

$$\begin{aligned} &\sum_{\omega \in \pi^{-1}(V_*)} \sum_{k=0}^{\infty} r_{w(k,\omega)}^{d_H - \sigma d_W} (A_{w(k,\omega)}(f) - f(\pi(\omega)))^2 \\ &\lesssim \sum_{\omega \in \pi^{-1}(V_*)} \sum_{k=0}^{\infty} r_{w(k,\omega)}^{d_H - \sigma d_W} (A_{w(k+1,\omega)}(f) - A_{w(k,\omega)}(f))^2. \end{aligned}$$

The lemma follows immediately from Lemma 7.6. \square

Lemma 7.8. *Let $\sigma > \frac{d_S}{2}$. For $f \in \tilde{\Gamma}^{\sigma}(K) \cap C(K)$, we have $f \in \Lambda_{\sigma}^{2,2}(K)$ with $\|f\|_{\Lambda_{\sigma}^{2,2}(K)} \lesssim \|f\|_{\tilde{\Gamma}^{\sigma}(K)}$.*

Proof. The lemma is true since

$$\begin{aligned}
\sum_{m=0}^{\infty} \lambda^{2m} \|\nabla_m f\|_{l^2(E_{v,m})}^2 &\asymp \sum_{m=0}^{\infty} \lambda^{2m} \sum_{w \in \Lambda_m} \sum_{p \neq q \in V_0} (f(F_w p) - f(F_w q))^2 \\
&\lesssim \sum_{m=0}^{\infty} \sum_{w \in \Lambda_m} r_w^{d_H - \sigma d_W} \sum_{p \in V_0} (A_w(f) - f(F_w p))^2 \\
&\lesssim \sum_{\omega \in \pi^{-1}(V_*)} \sum_{k=0}^{\infty} r_{w(k,\omega)}^{d_H - \sigma d_W} (A_{w(k,\omega)}(f) - f(\pi(\omega)))^2 \\
&\lesssim \|f\|_{\tilde{\Gamma}^\sigma(K)}^2,
\end{aligned}$$

where we use Lemma 7.7 in the last line. \square

Proof of Theorem 5.9. It is obvious that $\Gamma^\sigma(K) \subset \tilde{\Gamma}^\sigma(K)$ by Proposition 5.7. In addition, we can easily see $\Gamma^\sigma(K) \subset C(K), \forall \sigma > \frac{d_S}{2}$ by Theorem 6.7 and $\|S_l \tilde{f}_l\|_{L^\infty(K)} \lesssim \|D_l S_l \tilde{f}_l\|_{l^2(E_{c,l})}, \forall \tilde{f}_l \in \tilde{J}_l$. Thus, we have $\Gamma^\sigma(K) \subset \tilde{\Gamma}^\sigma(K) \cap C(K)$.

Conversely, by Lemma 7.8, we have $\tilde{\Gamma}^\sigma(K) \cap C(K) \subset \Lambda_\sigma^{2,2}(K)$ whenever $\sigma > \frac{d_S}{2}$. For $\frac{d_S}{2} < \sigma < 1$, this gives that $\tilde{\Gamma}^\sigma(K) \cap C(K) \subset \Gamma^\sigma(K)$ by Theorem 4.7 (a) and 6.1. For $\sigma \geq 1$, by the Remark after Definition 4.3, we then have $\tilde{\Gamma}^\sigma(K) \cap C(K) \subset \text{constants}$. The desired result follows. \square

8. ATOMIC DECOMPOSITIONS: HIGHER ORDERS

In this section, we discuss the proof of the atomic decomposition theorem (Theorem 4.1) of the spaces $H^\sigma(K)$ for $1 \leq \sigma < 2$. Since the story is parallel to the case $0 \leq \sigma < 1$, we will omit some details in the proof. As a byproduct, we will prove Theorem 4.7 (b).

As in previous sections, we write $\lambda = \lambda(\sigma) = r^{(d_H - \sigma d_W)/2}$.

8.1. Proof of $H^\sigma(K) = \tilde{\Lambda}_\sigma^{2,2}(K)$ with $\frac{d_S}{2} < \sigma < 2$. In this part, we develop necessary lemmas for the atomic decomposition for $1 \leq \sigma < 2$, analogous to Section 5 and 6. As an application, we will prove Theorem 4.7 (b).

Recall H_{Λ_m} defined in Definition 4.4. We have some easy estimates for tent functions concerning H_{Λ_m} .

Lemma 8.1. *For $m \geq l$ and $\varphi_l \in T_l$, we have*

$$\|H_{\Lambda_m} \varphi_l\|_{l^2(V_{\Lambda_m} \setminus V_0)} \asymp r^{-l} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})}.$$

Proof. First, we take $m = l$. It is not hard to see that

$$\|H_{\Lambda_l} \varphi_l\|_{l^2(V_{\Lambda_l} \setminus V_0)} \asymp r^{-l} \|\nabla_l \varphi_l\|_{l^2(E_{v,l})}.$$

Next, we claim that $\|H_{\Lambda_m} \varphi_l\|_{l^2(V_{\Lambda_m} \setminus V_0)} = \|H_{\Lambda_l} \varphi_l\|_{l^2(V_{\Lambda_l} \setminus V_0)}$ for $m \geq l$. To see this, we notice that the following equality holds for any $g \in \text{dom} \mathcal{E}$,

$$-(g, H_{\Lambda_l} \varphi_l) = \mathcal{E}_{\Lambda_l}(g, \varphi_l) = \mathcal{E}(g, \varphi_l) = \mathcal{E}_{\Lambda_m}(g, \varphi_l) = -(g, H_{\Lambda_m} \varphi_l).$$

This implies $H_{\Lambda_l} \varphi_l(x) = H_{\Lambda_m} \varphi_l(x), \forall x \in V_{\Lambda_l}$ and $H_{\Lambda_m} \varphi_l(x) = 0, \forall x \in V_{\Lambda_m} \setminus V_{\Lambda_l}$, as g can be arbitrarily chosen. \square

Using Lemma 8.1, by a similar proof of Theorem 5.8, we have

Lemma 8.2. *Let $\frac{d_S}{2} < \sigma < 2 - \frac{d_S}{2}$ and $f \in C(K)$. Write $f = \sum_{m=0}^{\infty} \varphi_m$ with $\varphi_0 \in \mathcal{H}_0$ and $\varphi_m \in T_m, \forall m \geq 1$. Then we have*

$$\|\varphi_0\|_{L^2(K)}^2 + \sum_{m=0}^{\infty} \lambda^{2m} \|\nabla_m \varphi_m\|_{l^2(E_{v,m})}^2 \asymp \|\varphi_0\|_{L^2(K)}^2 + \sum_{m=1}^{\infty} r^{2m} \lambda^{2m} \|H_{\Lambda_m} f\|_{l^2(V_{\Lambda_m} \setminus V_0)}^2.$$

Next, we introduce the following *smoothed tent functions*, analogous to smoothed Haar functions.

Definition 8.3. *For $m \geq 1$, for any $\varphi_m \in T_m$, define $\check{S}_m \varphi_m$ to be the unique function in $H^2(K)$ such that*

$$\check{S}_m \varphi_m(x) = \varphi_m(x), \forall x \in \bigcup_{w \in \Lambda_m} F_w V_0,$$

and

$$\|\Delta \check{S}_m \varphi_m\|_{L^2(K)} = \min\{\|\Delta f\|_{L^2(K)} : f \in H^2(K), f|_{V_{\Lambda_m}} = \varphi_m|_{V_{\Lambda_m}}\},$$

call it a *m-smoothed tent function*.

Define the space of *m-smoothed tent functions* by $\check{T}_m = \check{S}_m T_m$.

Similar to the argument below Definition 6.2, one can see that $\check{S}_m \varphi_m$ is unique and the map \check{S}_m is linear. This time, $H_D^2(K)$ is a Hilbert space, and for any $\varphi_m \in T_m$, the collection $\{f \in H^2(K), f|_{V_{\Lambda_m}} = \varphi_m|_{V_{\Lambda_m}}\}$ is a closed linear subspace of $H_D^2(K)$, so the minimum will be attained by some function $f \in H_D^2(K)$. In addition, by a standard variational argument, $\check{S}_m \varphi_m$ is uniquely characterized by

$$\int_K \Delta(\check{S}_m \varphi_m) \Delta g d\mu = 0,$$

for any $g \in H_D^2(K)$ such that $g|_{V_{\Lambda_m}} = 0$. This implies that \check{S}_m is a linear map.

In particular, we have the decomposition theorem (Theorem 8.4) analogous to Theorem 6.5 and 6.7.

Theorem 8.4. *For $\frac{d_S}{2} < \sigma \leq 2$, each function f in $H^\sigma(K)$ admits a unique series expansion $f = \sum_{m=0}^{\infty} \check{\varphi}_m$ with $\check{\varphi}_0 \in \mathcal{H}_0$ and $\check{\varphi}_m \in \check{T}_m$, and*

$$\|f\|_{H^\sigma(K)} \asymp (\|\check{\varphi}_0\|_{L^2(K)}^2 + \sum_{m=0}^{\infty} \lambda^{2m} \|\nabla_m \check{\varphi}_m\|_{l^2(E_{v,m})}^2)^{1/2}. \quad (8.1)$$

Moreover, for any series $f = \sum_{m=0}^{\infty} \check{\varphi}_m$, $f \in H^\sigma(K)$ if and only if

$$\sum_{m=0}^{\infty} \lambda^{2m} \|\nabla_m \check{\varphi}_m\|_{l^2(E_{v,m})}^2 < \infty.$$

The proof of Theorem 8.4 is very similar to that of Theorem 6.7. Below we list a key lemma but omit its proof.

Lemma 8.5. (a). *For $m \neq m' \geq 1$, we have $\Delta(\check{T}_m) \perp \Delta(\check{T}_{m'})$ in $L^2(K)$.*

(b). *For $m \geq 1$ and $f \in \mathcal{H}_0 \oplus (\oplus_{l=1}^m \check{T}_l)$, we have*

$$\|\Delta f\|_{L^2(K)} \asymp r^{-md_H/2} \|H_{\Lambda_m} f\|_{l^2(V_{\Lambda_m} \setminus V_0)}.$$

In particular, for any $m \geq 1$ and $\check{\varphi}_m \in \check{T}_m$, we have

$$\|\Delta \check{\varphi}_m\|_{L^2(K)} \asymp r^{-md_H/2} \|H_{\Lambda_m} \check{\varphi}_m\|_{l^2(V_{\Lambda_m} \setminus V_0)} \asymp r^{-m(1+d_H/2)} \|\nabla_m \check{\varphi}_m\|_{l^2(E_{v,m})}.$$

(c). For $m \geq 1$ and $f \in H^2(K)$, we have

$$\|H_{\Lambda_m} f\|_{l^2(V_{\Lambda_m} \setminus V_0)} \lesssim r^{md_H/2} \|\Delta f\|_{L^2(K)}.$$

In particular, for $m \geq l$ and $\check{\varphi}_l \in \check{T}_l$, we have

$$\|H_{\Lambda_m} \check{\varphi}_l\|_{l^2(V_{\Lambda_m} \setminus V_0)} \lesssim r^{md_H/2} \|\Delta \check{\varphi}_l\|_{L^2(K)} \asymp r^{md_H/2-l(1+d_H/2)} \|\nabla_l \check{\varphi}_l\|_{l^2(E_{v,l})}.$$

Using Lemma 8.5 (a) and (b), we get the decomposition of $H^2(K)$ analogous to Corollary 6.5. Using Lemma 8.5 (b) and (c), combining with Lemma 8.2 and the atomic decomposition Theorem for $\frac{d_S}{2} < \sigma < 1$, we get the decomposition of $H^\sigma(K)$ for $\frac{d_S}{2} < \sigma < 1$ analogous to Theorem 6.7. Then we finish the proof of Theorem 8.4 by using complex interpolation.

Proof of Theorem 4.7 (b). It follows easily from Theorem 8.4 and Lemma 8.5 (c). \square

8.2. Atomic decomposition: $1 \leq \sigma < 2$. The case $1 \leq \sigma < 2 - \frac{d_S}{2}$ follows immediately from Lemma 7.2, Lemma 8.2 and Theorem 4.7 (b). It remains to consider the case $2 - \frac{d_S}{2} < \sigma < 2$.

The proof relies on the following lemmas. Note that by Theorem 8.4, for $2 - \frac{d_S}{2} < \sigma < 2$, each $f \in H^\sigma(K)$ admits a unique expansion $f = \sum_{m=0}^{\infty} \check{\varphi}_m$ with $\check{\varphi}_0 \in \mathcal{H}_0$ and $\check{\varphi}_m \in \check{T}_m$, $\forall m \geq 1$.

Lemma 8.6. Let $l \geq 1$, $\check{\varphi}_l = GC_l + \sum_{m=1}^{\infty} G\tilde{f}_{m,l}$ with $C_l \in \mathbb{R}$ and $\tilde{f}_{m,l} \in \tilde{J}_m$.

(a). If $m < l$, then $\|\tilde{f}_{m,l}\|_{L^2(K)} \lesssim r^{(l-m)d_H/2-l(1+d_H/2)} \|\nabla_l \check{\varphi}_l\|_{l^2(E_{v,l})}$.

(b). If $m \geq l$, then $\|\tilde{f}_{m,l}\|_{L^2(K)} \lesssim r^{-l(1+d_H/2)} \|\nabla_l \check{\varphi}_l\|_{l^2(E_{v,l})}$.

In particular, $|C_l| \lesssim r^{-l} \|\nabla_l \check{\varphi}_l\|_{l^2(E_{v,l})}$.

Proof. Note that $-\Delta \check{\varphi}_l = C_l + \sum_{m=1}^{\infty} \tilde{f}_{m,l} \in L^2(K)$, and by Lemma 8.5, $\|\Delta \check{\varphi}_l\|_{L^2(K)} \asymp r^{-l(1+d_H/2)} \|\nabla_l \check{\varphi}_l\|_{l^2(E_{v,l})}$. Obviously, (b) is trivial. To prove (a), fix $w \in \Lambda_m$, and define a function ψ_w which takes value

$$\psi_w(x) = \begin{cases} 1, & \text{if } x \in V_{\Lambda_{l-1}} \cap (F_w K \setminus F_w V_0), \\ 0, & \text{if } x \in V_{\Lambda_{l-1}} \setminus (F_w K \setminus F_w V_0), \end{cases}$$

and is harmonic in each $F_{w'} K$, $w' \in \Lambda_{l-1}$. Then we have

$$\int_{F_w K} \Delta \check{\varphi}_l \cdot \psi_w d\mu = \int_K \Delta \check{\varphi}_l \cdot \psi_w d\mu = -\mathcal{E}(\check{\varphi}_l, \psi_w) = -\mathcal{E}_{\Lambda_{l-1}}(\check{\varphi}_l, \psi_w) = 0.$$

As a consequence, we have the estimate that

$$\begin{aligned} |A_w(\Delta \check{\varphi}_l)| &= |r_w^{-d_H} \int_{F_w K} \Delta \check{\varphi}_l d\mu| = |r_w^{-d_H} \int_{F_w K} (1 - \psi_w) \Delta \check{\varphi}_l d\mu| \\ &\lesssim r^{-md_H} r^{ld_H/2} \|\Delta \check{\varphi}_l\|_{L^2(F_w K)}, \end{aligned}$$

and thus

$$r^{md_H} |A_w(\Delta \check{\varphi}_l)|^2 \lesssim r^{(l-m)d_H} \|\Delta \check{\varphi}_l\|_{L^2(F_w K)}^2,$$

Thus (a) follows by summing up the above estimates over all cells $F_w K$ with $w \in \Lambda_m$ and using the fact $\|\Delta \check{\varphi}_l\|_{L^2(K)} \asymp r^{-l(1+d_H/2)} \|\nabla_l \check{\varphi}_l\|_{l^2(E_{v,l})}$. The estimate of $|C_l|$ follows by a same argument as that of (b). \square

Lemma 8.7. *Let $l \geq 1$, $\tilde{f}_l \in \tilde{J}_l$, write $G\tilde{f}_l = \sum_{m=1}^{\infty} \check{\varphi}_{m,l}$ with $\check{\varphi}_{m,l} \in \check{T}_m$.*

(a). *If $m < l$, then $\|\nabla_m \check{\varphi}_{m,l}\|_{l^2(E_{v,m})} \lesssim r^{(l-m)d_W/2+m(1+d_H/2)} \|\tilde{f}_l\|_{L^2(K)}$.*

(b). *If $m \geq l$, then $\|\nabla_m \check{\varphi}_{m,l}\|_{l^2(E_{v,m})} \lesssim r^{m(1+d_H/2)} \|\tilde{f}_l\|_{L^2(K)}$.*

Proof. By Theorem 8.4, the expansion $G\tilde{f}_l = \sum_{m=1}^{\infty} \check{\varphi}_{m,l}$ follows with

$$\|G\tilde{f}_l\|_{H^2(K)} \asymp \left(\sum_{m=1}^{\infty} r^{-2m(1+d_H/2)} \|\nabla_m \check{\varphi}_{m,l}\|_{l^2(E_{v,m})}^2 \right)^{1/2}.$$

This trivially gives (b). To prove (a), we first estimate $H_{\Lambda_m} G\tilde{f}_l$ for a fixed $m < l$. For each $x \in V_{\Lambda_m} \setminus V_0$, we have

$$H_{\Lambda_m} G\tilde{f}_l(x) = -\mathcal{E}(\psi_x^{(m)}, G\tilde{f}_l) = \int_K \psi_x^{(m)} \tilde{f}_l d\mu,$$

where $\psi_x^{(m)}$ is a piecewise harmonic function which takes value

$$\psi_x^{(m)}(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{if } x \in V_{\Lambda_m} \setminus \{x\}, \end{cases}$$

and is harmonic in each $F_w K$ with $w \in \Lambda_m$. Note that $\psi_x^{(m)} = \psi_x$ if $x \in V_{\Lambda_m} \setminus V_{\Lambda_{m-1}}$ in our previous notation for atomic decomposition theory. Denote $P_{\tilde{J}_l}$ the orthogonal projection from $L^2(K)$ onto \tilde{J}_l . We see that

$$\|P_{\tilde{J}_l} \psi_x^{(m)}\|_{L^2(K)} \asymp r^{ld_H/2} \|D_l P_{\tilde{J}_l} \psi_x^{(m)}\|_{l^2(E_{c,l})} \lesssim r^{ld_H/2} r^{l/2} \mathcal{E}^{1/2}(\psi_x^{(m)}) \lesssim r^{ld_H/2+(l-m)/2}.$$

Thus

$$\begin{aligned} |H_{\Lambda_m} G\tilde{f}_l(x)| &= \left| \int_K \psi_x^{(m)} \tilde{f}_l d\mu \right| = \left| \int_K P_{\tilde{J}_l} \psi_x^{(m)} \cdot \tilde{f}_l d\mu \right| \\ &\lesssim r^{ld_H/2+(l-m)/2} \|\tilde{f}_l\|_{L^2(\text{supp} P_{\tilde{J}_l} \psi_x^{(m)})}. \end{aligned}$$

Note that $P_{\tilde{J}_l} \psi_x^{(m)}$ is locally supported in $\bigcup \{F_w K : x \in F_w K, w \in \Lambda_m\}$. Summing the above estimate over vertices in $V_{\Lambda_m} \setminus V_0$, we get

$$\|H_{\Lambda_m} \sum_{m'=1}^m \check{\varphi}_{m',l}\|_{l^2(V_{\Lambda_m} \setminus V_0)} = \|H_{\Lambda_m} G\tilde{f}_l\|_{l^2(V_{\Lambda_m} \setminus V_0)} \lesssim r^{ld_H/2+(l-m)/2} \|\tilde{f}_l\|_{L^2(K)}.$$

Then by Lemma 8.5 (a) and (b), we get

$$\|\Delta \check{\varphi}_{m,l}\|_{L^2(K)} \lesssim \|\Delta \sum_{m'=1}^m \check{\varphi}_{m',l}\|_{L^2(K)} \lesssim r^{(l-m)d_W/2} \|\tilde{f}_l\|_{L^2(K)},$$

and thus by Lemma 8.5 (c), (a) follows immediately. \square

The proof of Theorem 4.1 for $2 - \frac{d_S}{2} < \sigma < 2$ is essentially the same as case for $\frac{d_S}{2} < \sigma < 1$ in Section 7, by using Lemma 8.6 and 8.7 instead of Lemma 7.3 and 7.4. Readers only need to carefully check the orders involved. We omit it.

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