SOBOLEV SPACES ON P.C.F. SELF-SIMILAR SETS II: BOUNDARY BEHAVIOR AND INTERPOLATION THEOREMS

SHIPING CAO AND HUA QIU

Abstract. We study the Sobolev spaces $H^\sigma(K)$ and $H^\sigma_0(K)$ on p.c.f. self-similar sets. First, for $\sigma \in \mathbb{R}^+$, we make an exact description of the tangents of functions in $H^\sigma(K)$ at the boundary, and introduce a countable set of critical orders that arises naturally in the boundary behavior of functions. These critical orders are just $\frac{1}{2}+\mathbb{Z}_+$ in the Euclidean case, but become complicated on fractals. Second, we characterize $H^\sigma_0(K)$ as the space of functions in $H^\sigma(K)$ with tangents of appropriate order, that depend on σ and critical orders, being 0. Last, we extend $H^\sigma(K)$ to $\sigma \in \mathbb{R}$, and obtain various interpolation theorems with $\sigma \in \mathbb{R}^+$ or $\sigma \in \mathbb{R}$. The interpolation space presents a critical phenomenon when the resulted order σ_θ is critical. Moreover, for the interpolation couple $(H^\sigma_0(K), H^{\sigma'}_0(K))$, more than the classical theorem, our interpolation theorem fully covers the teratological case that $\{\sigma, \sigma'\}$ contains at least one critical order.

Contents

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1. Introduction

The boundary behavior of functions, as an important topic in analysis on fractals, has been studied for years since the construction of the Laplacians on fractals. See [19, 20] for Kigami's construction of the Laplacians on p.c.f. self-similar sets, see [3, 4, 5, 14, 25, 26] for the probabilistic approach, and see books [21, 39] for further developments. Many important results are obtained, including tangents and gradients [23, 24, 31, 36, 37, 41, 10, 9, 12, 40]. See also [35, 36, 40] for related topics on the smooth bump functions, distributions and splines on fractals.

In this paper, we will take a further step to study the boundary behavior of functions in Sobolev spaces on p.c.f. self-similar sets, which are analogs of $H^\sigma(\Omega), \sigma\in\mathbb{R}$, in the $\mathbb{R}^n$ case. This is a continuation of our previous work [11].

Recall that for a domain Ω in $\mathbb{R}^n$ with smooth boundary, for $\sigma\in\mathbb{Z}_+$, $H^\sigma(\Omega)$ is the space of function f's on Ω such that f and its derivatives (in the sense of distributions) up to order σ are in $L^2(\Omega)$, and the definition of $H^\sigma(\Omega)$ can be generalized to all real σ via complex interpolation or other numerous equivalent methods. There are rich fundamental results concerning the boundary behavior of functions in $H^\sigma(\Omega)$, including the trace theorems, interpolation theorems, which provide powerful tools in the study of non-homogeneous boundary value problems and further topics on Ω. Among them, the characterization of $H^\sigma_0(\Omega)$, $\sigma\geq 0$,

$$H^\sigma_0(\Omega)=\big\{f\in H^\sigma(\Omega):\frac{\partial^j f}{\partial\nu^j}=0,\quad \forall 0\leq j<\sigma-\frac{1}{2}\big\}, \tag{1.1}$$

which was first discovered in [28, 29, 30] by J.L. Lions and E. Magenes, plays a central and delicate role. See monograph [27] for a systematic development and various applications.

We will reproduce the characterization (1.1) of $H^\sigma_0(\Omega)$ in the fractal setting, and stem from which, we aim to provide a throughout study on the interpolation of $H^\sigma(\Omega)$ on fractals. Due to the complicity of the fractal feature, we need to take a quite different approach.

In the classical setting, for $\sigma\notin\frac{1}{2}+\mathbb{Z}_+$, $H^\sigma_0(\mathbb{R}^n_+)$ consisting of Sobolev functions supported in $\mathbb{R}^n_+$, can be embedded into $H^\sigma(\mathbb{R}^n)$ as a subspace, by extending functions by 0 outside $\mathbb{R}^n_+$. What's more, there exists a retraction mapping $T: H^\sigma(\mathbb{R}^n)\to H^\sigma_0(\mathbb{R}^n_+)$. The proof of (1.1) and the interpolation result of $H^\sigma(\Omega)$ essentially rely on this extension, and the local coordinate representation of $H^\sigma(\Omega)$ along the boundary. The values in $\frac{1}{2}+\mathbb{Z}_+$ are called critical orders, since $H^\sigma_0(\Omega)$ will present some critical phenomena when σ is such a value.

However, on the p.c.f. self-similar sets, there are "derivatives" other than Laplacians and normal derivatives at the boundary, Although these new derivatives do not matter in the matching conditions when extending a function to a larger fractal domain, they indeed reflect the boundary behavior of functions [37]. As a consequence, for a p.c.f. fractal domain Ω with boundary, $H^\sigma_0(\Omega)$ is usually smaller than the space of functions in $H^\sigma(\tilde\Omega)$ with support in Ω, where $\tilde\Omega$ is a larger fractal domain, and the retraction mapping does not exist. In addition, the occurrence of the new derivatives will create more "critical orders" in the fractal setting.

Instead of extending functions by 0 outside the fractal, we will extract the information of the boundary behavior with a more straightforward method, by splitting the functions. See Lemma 4.4, Theorem 4.7 and the remark after Definition 5.5. This method features our work, and shows natural insight into the Sobolev spaces. Here we summary the decomposition of spaces by splitting, but leave the explanation of notations in later context.

Theorem 1. *Let K be a p.c.f. self-similar set with boundary V_0. Let $k\geq 1$ be an integer, and $0\leq\sigma\leq 2k$. We have*

$$H^\sigma(K\setminus V_0)=\ker_\sigma\mathcal{T}_{\mathcal{H}_{k-1}}\oplus\big(\oplus_{\omega\in\mathcal{P}}\mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(l^2(\mathcal{H}_{k-1},A_w;r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})\big)\big),$$

$$H_0^\sigma(K\setminus V_0)=\ker_\sigma\mathcal{T}_{\mathcal{H}_{k-1}}\oplus\big(\oplus_{\omega\in\mathcal{P}}\mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(\overline{l^2(\mathcal{H}_{k-1};r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})}\big)\big),$$

$$H_{00}^\sigma(K\setminus V_0)=\ker_\sigma\mathcal{T}_{\mathcal{H}_{k-1}}\oplus\big(\oplus_{\omega\in\mathcal{P}}\mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(l^2(\mathcal{H}_{k-1};r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})\big)\big).$$

A surprising consequence of the above decomposition result is that it provides privilege when considering the interpolation couple $(H_0^{\sigma_1}(K\setminus V_0),H_0^{\sigma_2}(K\setminus V_0))$ for at least one of σ_1,σ_2 being a critical order, while in the $\mathbb{R}^n$ case, Lions and Magenes's method will meet teratological difficulty, see [27] (Chapter 1, Section 18). For example, when K is chosen to be the unit interval $I=[0,1]$, we will have no difficulty to generalize the interpolation result for $[H^{\sigma_1}(0,1),H^{\sigma_2}(0,1)]_\theta$ when σ_1 or σ_2 is in $\frac{1}{2}+\mathbb{Z}_+$.

Now we briefly introduce our main results. Let K be a p.c.f. self-similar set which possesses a local regular Dirichlet form in the sense of Kigami. Let V_0 be its boundary consisting of finitely many points. For $\sigma\in\mathbb{R}$, by a slight abuse of notation, we write $H^\sigma(K)$ for the Sobolev space on the domain $K\setminus V_0$. A systematical introduction of Sobolev spaces can be found in [38] on fractals by R.S. Strichartz and in [15] on more general metric measure spaces by A. Grigor'yan. See also [11, 16, 17] for some equivalent Besov type characterizations of $H^\sigma(K)$, and [7, 8, 11, 13, 32] for related results on Besov spaces and interpolation properties. Our work deals with the boundaries of fractal domains, and interested readers can find some works on Euclidean domains with fractal boundaries [1, 33]. In this paper, we focus on the following three aspects.

Firstly, we study tangents at boundary points for functions in $H^\sigma(K)$ with $\sigma\in\mathbb{R}^+$. In history, various different approaches are developed towards gradients and tangents for functions on K. Typical ideas include defining the gradients by the energy measures [23, 24] and defining the tangents as the multiharmonic functions that match the local behavior of functions f at a generic point [41] or a vertex [37, 36] in K. We will introduce a simpler but more efficient description based on the latter idea, and give a thorough study of tangents at points in V_0 for functions in $H^\sigma(K)$. See Definition 3.3, Theorem 3.14 and Theorem 3.17.

Secondly, we study the Sobolev space $H_0^\sigma(K)$ with $\sigma\in\mathbb{R}^+$, which is defined as the closure of all compactly supported smooth functions with respect to the norm of $H^\sigma(K)$. In particular, analogously to (1.1), we will show that(Theorem 4.2),

Theorem 2. *For $\sigma\geq 0$, $H_0^\sigma(K)=\big\{f\in H^\sigma(K):T_\omega^{(\sigma)}(f)=0,\forall\omega\in\pi^{-1}(V_0)\big\}$. In particular, $H_0^\sigma(K)=H^\sigma(K)$ if $\sigma\leq\frac{d_S}{2}$.*

Here π is the canonical coding map associated with K, $T_\omega^{(\sigma)}(f)$(Definition 3.3 and 3.13) stands for the tangent of f at $\pi(\omega)$ with order that, roughly speaking, works best for $H^\sigma(K)$, and d_S is the spectral dimension of K. Readers are suggested to compare this result with the authors' previous work on the characterization of $H_D^\sigma(K)$ and $H_N^\sigma(K)$ in [11]. As we have mentioned, the proof of Theorem 2 essentially relies on the splitting method in Theorem 1, and is very different from Lions and Magenes's method for classical domains in [27]. We will use the smooth bump functions developed by L.G. Rogers, R.S. Strichartz and A. Teplyaev in [35] as an important tool.

Lastly, we study the interpolation theorems concerning $H^\sigma(K)$ and $H^\sigma_0(K)$. First, using the results obtained in the previous parts, we are ready to deal with the case $\sigma\in\mathbb{R}^+$(Theorem 5.4,5.6 and 5.7),

Theorem 3.

$$[H^\sigma(K),H^{\sigma'}(K)]_\theta=H^{(1-\theta)\sigma+\theta\sigma'}(K),\quad \forall\sigma>\sigma'\geq 0,\theta\in[0,1],$$

$$[H^\sigma_0(K),H^{\sigma'}_0(K)]_\theta=H^{(1-\theta)\sigma+\theta\sigma'}_{00}(K),\quad \forall\sigma>\sigma'\geq 0,\theta\in(0,1),$$

where $H^\sigma_{00}(K)\subset H^\sigma_0(K)$ are analogs of the Lions-Magenes spaces. In particular,

$$[H^\sigma_{00}(K),H^{\sigma'}_{00}(K)]_\theta=H^{(1-\theta)\sigma+\theta\sigma'}_{00}(K),\quad \forall\sigma>\sigma'\geq 0,\theta\in[0,1],$$

and $H^\sigma_{00}(K)=H^\sigma_0(K)$ except a countable set of critical orders of σ that arises naturally in Theorem 3.14 dealing with tangents of functions in $H^\sigma(K)$.

In particular, for H_0 type Sobolev spaces, this theorem covers the teratological case that σ or σ' is a critical order.

Moreover, we then introduce the space $H^\sigma(K)$ with $\sigma<0$ as the dual of $H^{-\sigma}_0(K)$, and extend the story of interpolation theorem to $\sigma\in\mathbb{R}$. The difficulty in this part lies in the fact that the domain of Laplacian is not generally closed under multiplication [6], and we develop a projection technique that preserves regularity instead. It holds that(Theorem 6.2),

Theorem 4. *For $-\infty<\sigma'<\sigma<\infty$ and $0<\theta<1$,*

$$[H^\sigma(K),H^{\sigma'}(K)]_\theta=\begin{cases}H^{\sigma_\theta}(K), & \text{if } \sigma_\theta=(1-\theta)\sigma+\theta\sigma'\geq 0,\\ \big(H^{-\sigma_\theta}_{00}(K)\big)', & \text{if } \sigma_\theta=(1-\theta)\sigma+\theta\sigma'<0.\end{cases}$$

In particular, $[H^\sigma(K),H^{\sigma'}(K)]_\theta=H^{\sigma_\theta}(K)$ except σ_θ is in the countable set of critical orders.

We briefly introduce the structure of our writing. In Section 2, we provide backgrounds and definitions that will be used later. Two conditions, (**C1**) and (**C2**), will be introduced for convenience. In Section 3, we study the tangents (and pre-tangents) at the boundary points for functions in $H^\sigma(K)$ with $\sigma\in\mathbb{R}^+$. In Section 4, we characterize $H^\sigma_0(K)$ in terms of the boundary behavior of functions in $H^\sigma(K)$. In Section 5, we develop interpolation theorems for Sobolev spaces $H^\sigma(K),H^\sigma_0(K)$ and $H^\sigma_{00}(K)$ with $\sigma\in\mathbb{R}^+$. In Section 6, we extend the interpolation theorem of $H^\sigma(K)$ to $\sigma\in\mathbb{R}$. In Section 7, we present some examples, along with some equivalent narrations of our results. In the Appendix, we present a useful decomposition lemma on certain weighted sequence spaces of functions with mixed norms, which plays a key role throughout the paper, then provide a brief discussion on how to prove the previous main theorems without assuming (**C1**).

Before ending this section, we point out some further investigations along the direction of this topic. It would be interesting to generalize the considerations to more complicated spaces, such as the Sierpinski carpet and more generally, Kigami's abstract resistance forms [22]. Another potential question is to utilize the results to study the resolvent to solutions of the resolvent equations on fractals, see [18, 34] for characterizations of the resolvent kernel of the Laplacian on p.c.f. self-similar sets and their blowups.

2. Preliminaries

We introduce some backgrounds in this section, including the Dirichlet forms and Sobolev spaces on p.c.f. self-similar sets.

Let $\{F_i\}_{i=1}^N$ be a finite collection of contractions on a complete metric space $(\mathcal{M}, d)$. The self-similar set associated with the *iterated function system (i.f.s.)* $\{F_i\}_{i=1}^N$ is the unique compact set $K \subset \mathcal{M}$ satisfying $K = \bigcup_{i=1}^N F_i K$. For $m \geq 1$, we define $W_m = \{1, \cdots, N\}^m$ the collection of *words* of length m, and for each $w = w_1 w_2 \cdots w_m \in W_m$, denote

$$F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}.$$

For uniformity, we set $W_0 = \{\emptyset\}$, with $F_\emptyset$ being the identity map. For convenience, let $W_* = \bigcup_{m=0}^\infty W_m$ be the collection of all finite words.

Let $\Sigma = \{1, 2, \cdots, N\}^{\mathbb{N}}$ be the shift space endowed with the natural product topology. There is a continuous surjection $\pi : \Sigma \to K$ defined by

$$\pi(\omega) = \bigcap_{m \geq 1} F_{[\omega]_m} K,$$

where for $\omega = \omega_1 \omega_2 \cdots$ in Σ we write $[\omega]_m = \omega_1 \omega_2 \cdots \omega_m \in W_m$ for each $m \geq 1$. Let

$$C_K = \bigcup_{i \neq j} F_i K \cap F_j K, \quad \mathcal{C} = \pi^{-1}(C_K), \quad \mathcal{P} = \bigcup_{m \geq 1} \sigma^m \mathcal{C},$$

where σ is the shift map define as $\sigma(\omega_1 \omega_2 \cdots) = \omega_2 \omega_3 \cdots$. $\mathcal{P}$ is called the *post critical set*. Call K a *p.c.f. self-similar set* if $\#\mathcal{P} < \infty$. In what follows, we always assume that K is a connected p.c.f. self-similar set. Let $V_0 = \pi(\mathcal{P})$ and call it the *boundary* of K. For $m \geq 1$, we always have $F_w K \cap F_{w'} K \subset F_w V_0 \cap F_{w'} V_0$ for any $w \neq w' \in W_m$.

We recall the fact that each $\omega \in \mathcal{P}$ takes the form $\omega = \tau \dot{w}$, with $\tau, w \in W_*$. For uniformity, we will use the same representation $\omega = \tau \dot{w}$ for each $\omega \in W_*$. In particular, we set $\tau = \emptyset$ if $\omega \in \mathcal{P}$ is periodic.

2.1. Dirichlet forms on p.c.f. self-similar sets.

Let's briefly recall the construction of Dirichlet forms on p.c.f. self-similar sets. Readers are suggested to refer to books [21] and [39] for any unexplained details and notations.

For $m \geq 1$, denote $V_m = \bigcup_{w \in W_m} F_w V_0$ and let $l(V_m) = \{f : f \text{ maps } V_m \text{ into } \mathbb{R}\}$. Write $V_* = \bigcup_{m \geq 0} V_m$.

Let $H = (H_{pq})_{p,q \in V_0}$ be a symmetric linear operator (matrix) on $l(V_0)$. H is called a *(discrete) Laplacian* on V_0 if H is non-positive definite; $Hu = 0$ if and only if u is constant on V_0; and $H_{pq} \geq 0$ for any $p \neq q \in V_0$. Given a Laplacian H on V_0 and a vector $\boldsymbol{r} = \{r_i\}_{i=1}^N$ with $r_i > 0$, $1 \leq i \leq N$, define the *(discrete) Dirichlet form* on V_0 by

$$\mathcal{E}_0(f, g) = -(f, Hg),$$

for $f, g \in l(V_0)$, and inductively on V_m by

$$\mathcal{E}_m(f, g) = \sum_{i=1}^N r_i^{-1} \mathcal{E}_{m-1}(f \circ F_i, g \circ F_i), m \geq 1,$$

for $f, g \in l(V_m)$. Write $\mathcal{E}_m(f) := \mathcal{E}_m(f, f)$ for short.

Say $(H, \boldsymbol{r})$ is a *harmonic structure* if for any $f \in l(V_0)$,

$$\mathcal{E}_0(f) = \min\{\mathcal{E}_1(g) : g \in l(V_1), g|_{V_0} = f\}.$$

In addition, call $(H,\boldsymbol{r})$ a *regular harmonic structure*, if $0<r_i<1,\forall 1\leq i\leq N$. In this paper, we will always assume that there exists a regular harmonic structure associated with K.

Now for each $f\in C(K)$, the sequence $\{\mathcal{E}_m(f)\}_{m\geq 0}$ is nondecreasing, so the following definitions make sense. Let $dom\mathcal{E}=\{f\in C(K):\lim_{m\to\infty}\mathcal{E}_m(f)<\infty\}$, and

$$\mathcal{E}(f,g)=\lim_{m\to\infty}\mathcal{E}_m(f,g) \text{ for } f,g\in dom\mathcal{E}.$$

We write $\mathcal{E}(f):=\mathcal{E}(f,f)$ for short, and call $\mathcal{E}(f)$ the *energy* of f. It is known that $(\mathcal{E},dom\mathcal{E})$ turns out to be a *local regular Dirichlet form* on $L^2(K,\mu)$ for any Radon measure μ on K.

An important feature of the form $(\mathcal{E},dom\mathcal{E})$ is the following *self-similar identity*,

$$\mathcal{E}(f,g)=\sum_{i=1}^N r_i^{-1}\mathcal{E}(f\circ F_i,g\circ F_i),\quad \forall f,g\in dom\mathcal{E}. \tag{2.1}$$

Denote $r_w=r_{w_1}r_{w_2}\cdots r_{w_m}$ for each $w\in W_m,m\geq 0$. Then for $m\geq 1$, we have

$$\mathcal{E}_m(f,g)=\sum_{w\in W_m}r_w^{-1}\mathcal{E}_0(f\circ F_w,g\circ F_w),\quad \mathcal{E}(f,g)=\sum_{w\in W_m}r_w^{-1}\mathcal{E}(f\circ F_w,g\circ F_w).$$

Lastly, we need to mention that there is a natural metric on K related with the energy form $(\mathcal{E},dom\mathcal{E})$, called the *effective resistance metric*, which is defined as

$$R(x,y)=\big(\min\{\mathcal{E}(f):f\in dom\mathcal{E} \text{ and } f(x)=1,f(y=0)\}\big)^{-1},\forall x\neq y\in K.$$

2.2. **The Laplacians and Sobolev spaces.** Now we come to the basic concepts of the Laplacians and Sobolev spaces on K. Readers may find detailed backgrounds and further discussions in various contexts, for example [11, 21, 38, 39].

We always choose μ to be a self-similar measure on K in this paper. To be more precise, we fix a weight vector $\{\mu_i\}_{i=1}^N$, and let μ be the unique probability measure supported on K such that

$$\mu(A)=\sum_{i=1}^N\mu_i\mu(F_i^{-1}A),\quad \forall A\subset K.$$

One can easily check that $\mu(F_wK)=\mu_w:=\mu_{w_1}\cdots\mu_{w_m}$, for each $w\in W_m$.

For $f\in dom\mathcal{E}$, say $\Delta_\mu f=u$ if

$$\mathcal{E}(f,\varphi)=-\int_K u\varphi d\mu$$

holds for any $\varphi\in dom_0\mathcal{E}$, with $dom_0\mathcal{E}=\{\varphi\in dom\mathcal{E}:\varphi|_{V_0}=0\}$.

Write $\Delta_\mu=\Delta$ and $L^2(K,\mu)=L^2(K)$ for short.

Definition 2.1. *For $k\in\mathbb{Z}_+$, define the Sobolev space $H^{2k}(K)$ as*

$$H^{2k}(K)=\{f\in L^2(K):\Delta^j f\in L^2(K) \text{ for all } 0\leq j\leq k\}$$

with the norm $\|f\|_{H^{2k}(K)}$ of f given by

$$\|f\|^2_{H^{2k}(K)}=\sum_{j=0}^k\|\Delta^j f\|^2_{L^2(K)}\asymp\|f\|^2_{L^2(K)}+\|\Delta^k f\|^2_{L^2(K)}.$$

For $0<\theta<1$, $k\in\mathbb{Z}_+$, define $H^{2k+2\theta}(K)$ to be the complex interpolation space

$$H^{2(k+\theta)}(K)=[H^{2k}(K),H^{2k+2}(K)]_\theta.$$

Analogously, by additionally requiring that each $\Delta^j f$ satisfies the Dirichlet boundary condition for $j<k$ in the above definition when $k\in\mathbb{Z}_+$, we get a subspace, denoted by $H^{2k}_D(K)$, of $H^{2k}(K)$. The definition can be extended to any $\sigma\geq 0$ by using Bessel type potentials. For $\sigma\geq 0$, we have $H^\sigma_D(K)=(id-\Delta_D)^{-\sigma/2}L^2(K)$, with norm $\|(id-\Delta_D)^{\sigma/2}f\|_{L^2(K)}$, where Δ_D is the Dirichlet Laplacian. In particular, for $k\in\mathbb{Z}_+$ and $f\in H^{2k}_D(K)$, we have

$$\|f\|_{H^{2k}(K)}\asymp\|f\|_{H^{2k}_D(K)}\asymp\|\Delta^k f\|_{L^2(K)}.$$

Similarly, we can define $H^\sigma_N(K)=(id-\Delta_N)^{-\sigma/2}L^2(K)$ with Δ_N being the Neumann Laplacian. See [38] by Strichartz for more details.

Before the end of this section, we mention two conditions that we will assume throughout the main part of this paper (Section 3 to Section 6).
(**C1**): *For any $p\in V_0$, we assume $\#\pi^{-1}(p)=1$.*
(**C2**): *There exists $d_H>0$ such that $\mu_i=r_i^{d_H}$, $\forall 1\leq i\leq N$.*

The condition (**C1**) is a geometric condition, which is naturally satisfied for nested fractals, see [21, 26]. When (**C1**) holds, we only have one outward "direction" for each boundary vertex, so we can avoid the tedious treatments of the "matching condition" on the boundary. However, **all the main theorems** in this paper, including Theorem 3.14, 3.17, 4.2, 5.4, 5.6, 5.7 and 6.2, are valid even if (**C1**) is not satisfied, as long as we assume (**C2**). See Appendix B for a further discussion on (**C1**).

The condition (**C2**) means that the self-similar measure μ is d_H-regular with respect to the effective resistance metric R. In fact, d_H is uniquely determined by the equation $\sum_{i=1}^N r_i^{d_H}=1$ and μ is uniquely determined by (**C2**). To be precise, for $x\in K$, $\rho>0$, denote $B_R(x,\rho)=\{y\in K: R(x,y)<\rho\}$ the *ball* centered at x with radius ρ. It is well-known that the measure μ under condition (**C2**) is comparable with R, i.e. $\mu\big(B_R(x,\rho)\big)\asymp\rho^{d_H}$, see [2, Proposition 8.9]. Thus indeed d_H is the Hausdorff dimension of K with respect to the resistance metric R. With (**C2**) assumed, we have some clear descriptions of Sobolev spaces of lower orders σ, see [11, 17]. In particular, most theorems in this paper will fail for lower σ without (**C2**).

Throughout the paper, we always write $f\lesssim g$ if $f\leq Cg$ for some constant $C>0$, and write $f\asymp g$ if both $f\lesssim g$ and $g\lesssim f$. In addition, we write $X=\oplus_{k=1}^n X_k$ for Banach spaces X and $X_k, 1\leq k\leq n$, if
1. $X_k\subset X$ and $\|\cdot\|_{X_k}\asymp\|\cdot\|_X$, for each $1\leq k\leq n$;
2. for each $x\in X$, there is a unique representation $x=\sum_{k=1}^n x_k$, with $x_k\in X_k, 1\leq k\leq n$.

3. Boundary behaviors of $H^\sigma(K)$

In this section, we study the boundary behavior of functions in Sobolev spaces $H^\sigma(K)$, $\sigma\geq 0$. We focus on the tangents of functions at the boundary V_0, and introduce the concept of pre-tangents, which will serve as an important tool in further development throughout Section 4-6, including the characterization theorem of $H^\sigma_0(K)$, and interpolation theorems of various Sobolev spaces. We assume the condition (**C2**) in this section.

3.1. **Tangents and pre-tangents.** For a function f on $\mathbb{R}$, we have the Taylor series expansion

$$f(x)=f(x_0)+f'(x)(x-x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2+\cdots.$$

The polynomial $f(x_0)+f'(x)(x-x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$ which serves as a tangent of f at x_0, reflects the local behavior of f, and includes all its information of derivatives with orders no more than n at x_0. Analogously, it is natural to define tangents of functions on K at a point as elements of multiharmonic functions (Definition 3.1). We will focus on boundary vertices in this paper, and our definition of tangents (Definition 3.3) is modified from that of Rogers and Strichartz [36, 37]. Readers can also find theorems on tangents at generic points in [41].

Definition 3.1. *For $k\geq 0$, let $\mathcal{H}_k=\{f\in H^{2k+2}(K):\Delta^{k+1}f=0\}$ be the space of $(k+1)$-multiharmonic functions on K. Let $\mathcal{H}_\bullet=\bigcup_{k=0}^\infty\mathcal{H}_k$.*

We consider a sequence of cells that shrink to a boundary vertex, and order multiharmonic functions according to the convergence rate, analogous to $(x-x_0)^n$ in the classical cases.

Definition 3.2. *Fix $w\in W_*$.*

(a). Define A_w by $A_wf(x)=f(F_wx)$ for any function f on K.

(b). Let $\{\lambda_{l,w}\}_{l=0}^\infty$ be the set of nonzero eigenvalues of $A_w:\mathcal{H}_\bullet\to\mathcal{H}_\bullet$, which is ordered in decreasing order of absolute values, i.e. $1=|\lambda_{0,w}|\geq|\lambda_{1,w}|\geq|\lambda_{2,w}|\geq\cdots$. Let $E_{l,w}=\bigcup_{n=1}^\infty\ker(A_w-\lambda_{l,w})^n\subset\mathcal{H}_\bullet$ be the generalized eigenspace of A_w corresponding to $\lambda_{l,w}$.

(c). Define $\tilde{E}_{l,w}=\oplus_{i=0}^{l''}E_{i,w}$ and $\hat{E}_{l,w}=\oplus_{i=l'}^{l''}E_{i,w}$, where

$$l'=\min\{i\geq 0:|\lambda_{i,w}|=|\lambda_{l,w}|\},\quad l''=\max\{i\geq 0:|\lambda_{i,w}|=|\lambda_{l,w}|\}.$$

Remark. (a). It is well known that $\lim_{l\to\infty}|\lambda_{l,w}|=0$, and $E_{l,w}$ is of finite dimension for each l. In addition, we have $\lambda_{0,w}=1$ and $E_{0,w}=\hat{E}_{0,w}=\tilde{E}_{0,w}=$ *constants*.

(b). If $|\lambda_{l_1,w}|=|\lambda_{l_2,w}|$, then $\tilde{E}_{l_1,w}=\tilde{E}_{l_2,w}$ and $\hat{E}_{l_1,w}=\hat{E}_{l_2,w}$. In other words, the definition of $\tilde{E}_{l,w}$ and $\hat{E}_{l,w}$ only depends on the absolute value of $\lambda_{l,w}$.

Now, we define the tangent of a function f at a boundary vertex $p\in V_0$.

Definition 3.3. *Let $f\in C(K)$, $\omega=\tau\dot{w}\in\mathcal{P}$ with $\dot{w}=ww\cdots$, and $l\geq 0$. A multiharmonic function $h\in\tilde{E}_{l,w}$ is called a l-tangent of f at $\pi(\omega)$ if*

$$\|A_\tau f-h\|_{L^\infty(F_w^nK)}=o(\lambda_{l,w}^n),$$

and we denote $T_{l,\omega}f:=h$. In particular, $\|f-T_{l,\omega}f\|_{L^\infty(F_w^nK)}=o(\lambda_{l,w}^n)$ if $\tau=\emptyset$.

Intuitively, each $\omega\in\mathcal{P}$ represents a boundary point $p\in V_0$ and a "direction" that approaches p. We can view the collection of tangents $T_{l,\omega},\omega\in\pi^{-1}(p)$ as the "tangent" at the boundary point $p\in V_0$.

On the other hand, for further development in Section 4-6, we need more delicate descriptions of the boundary behavior of f. For this purpose, we record some important information in a sequence of multiharmonic functions that reflects the "average" of f on each cell.

Notation. *Let X be a closed subspace of $L^2(K)$, and $w\in W_*$.*

(a). Define $A_{w,X}=A_w\circ P_X$, where P_X is the orthogonal projection from $L^2(K)$ onto X.

(b). Define $\mathcal{A}_w(f)=\{A_w^nf\}_{n\geq 0}$ for each $f\in L^2(K)$.

(c). With little abuse of notations, we write $P_X\mathcal{A}_w(f)=\{P_XA_w^nf\}_{n\geq 0}$.

Definition 3.4. *Fix $l\geq 0$ such that for any distinct $\omega=\tau\dot{w}$ and $\omega'=\tau'\dot{w}'$ in $\mathcal{P}$, we have $F_\tau F_w^lK\cap F_{\tau'}F_{w'}^lK=\emptyset$. Let $k\in\mathbb{Z}^+$, $\omega=\tau\dot{w}\in\mathcal{P}$, f be a function on K. Define $\mathcal{T}_{\mathcal{H}_{k-1},\omega}f=P_{\mathcal{H}_{k-1}}\mathcal{A}_w(A_w^lA_\tau f)$. Call $\mathcal{T}_{\mathcal{H}_{k-1},\omega}f$ the k-pre-tangent of f at $\pi(\omega)$.*

Remark. In Definition 3.4, the requirement $F_\tau F_w^l K\cap F_{\tau'}F_{w'}^l K=\emptyset$ ensures that the pre-tangents of different vertices will not interfere each other. This will provide some convenience in Section 4 and 5.

Before ending this subsection of definitions, we would like to introduce some sequence spaces which will frequently occur.

Definition 3.5. *Let X be a Banach space, with norm $\|\cdot\|_X$, and $A:X\to X$ be a compact operator. Denote $\sigma(A,X)$ the spectrum of $A:X\to X$.*

(a). For $\alpha>0$, define

$$l^2(X;\alpha)=\big\{\boldsymbol{s}=\{s_n\}_{n=0}^\infty:\{\alpha^{-n}\|s_n\|_X\}_{n=0}^\infty\in l^2\big\},$$

with norm $\|\boldsymbol{s}\|_{l^2(X;\alpha)}=\big\|\alpha^{-n}\|s_n\|_X\big\|_{l^2}$.

(b). For $\alpha>0$, define

$$l^2(X,A;\alpha)=\big\{\boldsymbol{s}=\{s_n\}_{n=0}^\infty:\{s_{n+1}-As_n\}_{n=0}^\infty\in l^2(X;\alpha)\big\},$$

with norm $\|\boldsymbol{s}\|_{l^2(X,A;\alpha)}=\|s_{n+1}-As_n\|_{l^2(X;\alpha)}+\|s_0\|_X$.

(c). For each $s\in X$, define $\mathcal{S}_A(s)=\{A^ns\}_{n=0}^\infty$, with norm $\|\mathcal{S}_A(\cdot)\|_{\mathcal{S}_A(X)}=\|\cdot\|_X$.

In applications, we usually take X to be certain subspaces of $L^2(K)$, and A to be A_w or $A_{w,X}$.

We have a lemma (Lemma 3.6) connecting the above three classes of sequence spaces, which will be used frequently in this paper. We take the same setting as in Definition 3.2. Let $\{\lambda_l\}_{l\geq 0}$ be the nonzero eigenvalues of A, which is ordered in decreasing order of absolute values. Let E_l be the corresponding generalized eigenspaces. In addition, denote $\tilde{E}_l=\oplus_{i=0}^{l''}E_i$ and $\hat{E}_l=\oplus_{i=l'}^{l''}E_i$, where $l'=\min\{i\geq 0:|\lambda_i|=|\lambda_l|\}, l''=\max\{i\geq 0:|\lambda_i|=|\lambda_l|\}$.

Lemma 3.6. *Let $\overline{l^2(X;\alpha)}$ be the closure of $l^2(X;\alpha)$ in $l^2(X,A;\alpha)$. Then $\overline{l^2(X;\alpha)}=l^2(X;\alpha)$ if and only if $\alpha\notin\{|\lambda_l|\}_{l\geq 0}$. In addition,*

(a). For $\alpha\geq|\lambda_0|$ or $\sigma(A,X)=\{0\}$, we have $l^2(X,A;\alpha)=\overline{l^2(X;\alpha)}$.

(b). For $|\lambda_{l+1}|\leq\alpha<|\lambda_l|$ or $\alpha<|\lambda_l|=\min\{|\lambda_k|:\lambda_k\in\sigma(A,X)\}$, we have

$$l^2(X,A;\alpha)=\mathcal{S}_A(\tilde{E}_l)\oplus\overline{l^2(X;\alpha)}.$$

Remark. We arrange the proof of this lemma in Appendix A. In the rest of this paper, without further clarification, we will always take $\overline{l^2(X;\alpha)}$ to be the closure of $l^2(X;\alpha)$ in $l^2(X,A;\alpha)$ as in the lemma.

3.2. **Construction of tangents.** In this subsection, we construct the tangents for functions $f\in H^\sigma(K)$ at points in V_0. We treat the higher order case ($\sigma\geq 2$) and the lower order case ($\sigma<2$) separately. For simplicity, we will fix an $\omega=\dot{w}\in\mathcal{P}$.

Higher order case. We start by studying functions in $H^{2k}(K), k\geq 1$.

Lemma 3.7. *Let $k\geq 1$. There exists $g_{w,k}\in C(K\times K)$ such that*

$$A_wf(x)-A_{w,\mathcal{H}_{k-1}}f(x)=\int_K g_{w,k}(x,y)(-\Delta)^kf(y)d\mu(y)$$

for any $f\in H^{2k}(K)$.

Proof. It is easy to see that $(A_w - A_{w,\mathcal{H}_{k-1}})|_{\mathcal{H}_{k-1}} = 0$. So it suffices to prove the lemma for $f = G^k(-\Delta)^k f$, where G is the Green's operator. We only need to take

$$g_{w,k}(x,y) = \int_{K^{k-1}} (A_w - A_{w,\mathcal{H}_{k-1}}) G_{y_1}(x) G(y_1,y_2)\cdots G(y_{k-1},y) d\mu(y_1)\cdots d\mu(y_{k-1}),$$

where $G_y(x) = G(x,y)$ is the Green's function. □

Lemma 3.8. *Let $f \in L^2(K)$ and $g \in L^\infty(K)$. Define $\boldsymbol{s} = \{\mu_w^{n/2} \int_K A_w^n f(x) g(x) d\mu(x)\}_{n=0}^\infty$, then we have*

$$\|\boldsymbol{s}\|_{l^2} \lesssim \|f\|_{L^2(K)} \|g\|_{L^\infty(K)}.$$

Proof. Let $Z = K \setminus F_w K$. Then $\|f\|_{L^2(K)} = \big\| \|\mu_w^{n/2} A_w^n f\|_{L^2(Z)} \big\|_{l^2}$ by scaling, and

$$\begin{aligned}
\Big|\int_K f(x)g(x)d\mu(x)\Big| &= \Big|\sum_{m=0}^\infty \mu_w^m \int_Z A_w^m f(x) A_w^m g(x) d\mu(x)\Big| \\
&\leq \sum_{m=0}^\infty \mu_w^m \|A_w^m f\|_{L^2(Z)} \|g\|_{L^\infty(K)}.
\end{aligned}$$

So using Minkowski inequality, we get

$$\begin{aligned}
\|\boldsymbol{s}\|_{l^2} &\leq \|g\|_{L^\infty(K)} \Big\| \sum_{m=0}^\infty \mu_w^{n/2} \mu_w^m \|A_w^{n+m} f\|_{L^2(Z)} \Big\|_{l^2} \\
&\leq \|g\|_{L^\infty(K)} \sum_{m=0}^\infty \mu_w^{m/2} \|f\|_{L^2(K)}.
\end{aligned}$$

Since $\mu_w < 1$, we get the lemma. □

Using Lemma 3.7 and 3.8, we get the following key observation.

Proposition 3.9. $\mathcal{A}_w \in \mathcal{L}\Big(H^\sigma(K), l^2\big(L^\infty(K), A_{w,\mathcal{H}_{k-1}}; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}\big)\Big)$ *for $\sigma \geq 2$ and $k \geq \lceil \sigma/2 \rceil$.*

Proof. First, we consider the $H^{2k}(K), k \geq 1$ case. We have the estimate

$$\begin{aligned}
\|A_w^{n+1} f - A_{w,\mathcal{H}_{k-1}} A_w^n f\|_{L^\infty(K)} &\leq \Big| \int_K \|g_{w,k}(\cdot,y)\|_{L^\infty(K)} (-\Delta)^k (A_w^n f)(y) d\mu(y) \Big| \\
&= \Big| r_w^{kn} \mu_w^{kn} \int_K \|g_{w,k}(\cdot,y)\|_{L^\infty(K)} A_w^n\big((-\Delta)^k f\big)(y) d\mu(y) \Big|,
\end{aligned}$$

by using Lemma 3.7 and scaling. Since $\|\Delta^k f\|_{L^2(K)} \leq \|f\|_{H^{2k}(K)}$, by using Lemma 3.8, we have proved the assertion for the $H^{2k}(K)$ case.

For general case, we need to use the complex interpolation. For any $k' \geq k$, we see that

$$\begin{aligned}
l^2\big(L^\infty(K), A_{w,\mathcal{H}_{k-1}}; r_w^k \mu_w^{k-1/2}\big) &= l^2\big(\mathcal{H}_{k-1}, A_{w,\mathcal{H}_{k-1}}; r_w^k \mu_w^{k-1/2}\big) + l^2(L^\infty(K); r_w^k \mu_w^{k-1/2}) \\
&= l^2\big(\mathcal{H}_{k-1}, A_{w,\mathcal{H}_{k'}}; r_w^k \mu_w^{k-1/2}\big) + l^2(L^\infty(K); r_w^k \mu_w^{k-1/2}) \\
&= l^2\big(L^\infty(K), A_{w,\mathcal{H}_{k'}}; r_w^k \mu_w^{k-1/2}\big),
\end{aligned}$$

where the first and last equalities are easy consequences of Lemma 3.6, using the fact that $\tilde{E}_{l,w} \subset \mathcal{H}_{k-1}$ for the largest l with $|\lambda_{l,w}| \geq r_w^k \mu_w^{k-1/2}$.

The proposition then follows from the fact that

$$[l^2(X,A;\alpha),l^2(X,A;\beta)]_\theta = l^2(X,A;\alpha^{(1-\theta)}\beta^\theta),$$

and $[H^{2k}(K),H^{2k+2}(K)]_\theta = H^{2k+2\theta}(K)$ with $\theta\in[0,1]$. □

Using Lemma 3.6, we get the existence of tangents for functions in $H^\sigma(K)$ with higher orders.

Lower order case. For this case, we need condition (**C2**), which guarantees that $H^\sigma(K)\subset L^\infty(K)$ as long as $\sigma>\frac{d_S}{2}$, where $d_S=\frac{2d_H}{1+d_H}$ is the *spectral dimension* of K. When K is the unit interval, we have $\frac{d_S}{2}=\frac{1}{2}$, which is indeed the critical order in the Euclidean case (See [27]).

For $0<t\leq 1$, define

$$\Lambda(t)=\{u\in W_*: r_u\leq t< r_{u^*}\},$$

where for $u=u_1u_2\cdots u_m$, $u^*=u_1u_2\cdots u_{m-1}$. In particular, set $r=\min_{i=1}^N r_i$, and let $\Lambda_m=\Lambda(r^m)$ for $m\geq 0$. For any $u\in W_*$, define the average of f on F_uK by $Avg_u(f)=\mu_u^{-1}\int_{F_uK} fd\mu$. In particular, $Avg_\emptyset(f)=\int_K fd\mu$.

For $m\geq 1$, define the space of *m-Haar functions*

$$\tilde{J}_m=\{\tilde{f}_m=\sum_{u\in\Lambda_m} c_u 1_{F_uK}: c_u\in\mathbb{R}, Avg_{u'}(\tilde{f}_m)=0,\forall u'\in\Lambda_{m-1}\},$$

where 1_E is the *characteristic function* of a set E. Let J_0 be the space of constant functions. Let

$$\tilde{\Gamma}_\sigma(K)=\big\{f\in L^2(K):\{P_{\tilde{J}_n}f\}_{n=0}^\infty\in l^2(L^2(K);r^{\sigma(1+d_H)/2})\big\},$$

with norm $\|\cdot\|_{\tilde{\Gamma}_\sigma(K)}=\|\{P_{\tilde{J}_n}\cdot\}_{n=0}^\infty\|_{l^2(L^2(K);r^{\sigma(1+d_H)/2})}$. The following result comes from Theorem 5.9 and 6.1 in [11].

Proposition 3.10. *For $\frac{d_S}{2}<\sigma<1$, we have $H^\sigma(K)=\tilde{\Gamma}_\sigma(K)\cap C(K)$, with $\|f\|_{H^\sigma(K)}\asymp\|f\|_{\tilde{\Gamma}_\sigma(K)}$.*

Using Proposition 3.10, we get the following estimate.

Lemma 3.11. *For each $f\in H^\sigma(K)$ with $\frac{d_S}{2}<\sigma<1$, let $f_n=\sum_{m=n}^\infty P_{\tilde{J}_m}f$. Then, we have $\{f_n\}_{n\geq 0}\in l^2(L^\infty(K);r^{\sigma/2}r^{(\sigma-1)d_H/2})$.*

Proof. Noticing that $\|P_{\tilde{J}_n}f\|_{L^\infty(K)}\lesssim r^{-nd_H/2}\|P_{\tilde{J}_n}f\|_{L^2(K)}$, by using Minkowski inequality, we have

$$\big\|r^{-n\sigma/2}r^{n(1-\sigma)d_H/2}\|f_n\|_{L^\infty(K)}\big\|_{l^2}\lesssim\big\|r^{-n\sigma/2}r^{n(1-\sigma)d_H/2}\sum_{m=0}^\infty r^{-(m+n)d_H/2}\|P_{\tilde{J}_{m+n}}f\|_{L^2(K)}\big\|_{l^2}$$

$$=\|\sum_{m=0}^\infty r^{m\sigma/2}r^{m(\sigma-1)d_H/2}r^{-(m+n)\sigma(1+d_H)/2}\|P_{\tilde{J}_{m+n}}f\|_{L^2(K)}\big\|_{l^2}$$

$$\lesssim\sum_{m=0}^\infty r^{m\sigma/2}r^{m(\sigma-1)d_H/2}\|f\|_{H^\sigma(K)}.$$

This finishes the proof. □

Using Lemma 3.11, we can easily get the following proposition.

Proposition 3.12. *$\mathcal{A}_w \in \mathcal{L}\Big(H^\sigma(K), l^2\big(L^\infty(K), A_{w,\mathcal{H}_{k-1}}; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}\big)\Big)$ for $\sigma > \frac{d_S}{2}$ and $k \geq \lceil \sigma/2 \rceil$.*

Proof. By Proposition 3.9, it suffices to prove the case $\sigma < 2$. For $\frac{d_S}{2} < \sigma < 1$, it is easy to see that for $f \in H^\sigma(K)$ and $n \geq 0$, choosing l such that $r^{l+1} < r_w^n \leq r^l$, we have

$$\|A_w^{n+1}f - A_w P_{\tilde{J}_0} A_w^n f\|_{L^\infty(K)} \leq \|A_w^n f - P_{\tilde{J}_0} A_w^n f\|_{L^\infty(K)} \lesssim \|\sum_{m=l+1}^\infty P_{\tilde{J}_m} f\|_{L^\infty(K)}.$$

It then follows from Lemma 3.11 that

$$\{A_w^n f\}_{n\geq 0} \in l^2(L^\infty(K), A_w P_{\tilde{J}_0}; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}).$$

Using the well-known fact that $E_{0,w} = \tilde{J}_0$, the fact that $\lambda_{1,w} = r_w < r_w^{\sigma/2}\mu_w^{(\sigma-1)/2} < 1$ and Lemma 3.6 (b), we see that

$$l^2(L^\infty(K), A_w P_{\tilde{J}_0}; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}) = l^2(L^\infty(K), A_{w,\mathcal{H}_0}; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}).$$

This implies that

$$\mathcal{A}_w \in \mathcal{L}\Big(H^\sigma(K), l^2\big(L^\infty(K), A_{w,\mathcal{H}_0}; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}\big)\Big).$$

The rest of the proof follows from complex interpolation, and using Proposition 3.9. □

A theorem on tangents. Now, we conclude our results in the following theorem. For convenience, in the rest of this paper, we use the following notations.

Definition 3.13. *For $\sigma > d_S/2$ and $\omega = \tau\dot{w} \in \mathcal{P}$, let $l_\omega(\sigma)$ be the unique integer such that*

$$|\lambda_{l_\omega(\sigma)+1,w}| \leq r_w^{\sigma/2}\mu_w^{(\sigma-1)/2} < |\lambda_{l_\omega(\sigma),w}|.$$

For convenience, we write $T_\omega^{(\sigma)} = T_{l_\omega(\sigma),\omega}$ for short.

In particular, when $\sigma \leq d_S/2$, we let $l_\omega(\sigma) = -1$ and $T_{l_\omega(\sigma),\omega} = 0$. Also, set $\mathcal{H}_{-1} = \{0\}$ for convenience.

From Proposition 3.9, Proposition 3.12 and Lemma 3.6, we have

Theorem 3.14. *Let $\omega = \tau\dot{w} \in \mathcal{P}$ and $\sigma > d_S/2$, then $A_\tau T_\omega^{(\sigma)} \in \mathcal{L}(H^\sigma(K), \tilde{E}_{l_\omega(\sigma),w})$. In addition,*

$$\sum_{n=0}^\infty r_w^{-\sigma n}\mu_w^{(1-\sigma)n}\|A_\tau f - T_\omega^{(\sigma)} f\|^2_{L^\infty(F_w^n K)} \lesssim \|f\|^2_{H^\sigma(K)},$$

if $|\lambda_{l_\omega(\sigma)+1,w}| < r_w^{\sigma/2}\mu_w^{(\sigma-1)/2} < |\lambda_{l_\omega(\sigma),w}|$.

Remark 1. Theorem 3.14 is still true for $\sigma \geq 2$ without condition (**C2**).
Remark 2. The estimate in Theorem 3.14 is sharp, which can be deduced from a stronger result (Theorem 3.17).

3.3. **A theorem on pre-tangents.** In the rest of this section, we focus on pre-tangents. For convenience, in the discussion below, we fix $w \in W_*$ and consider $P_{\mathcal{H}_{k-1}}\mathcal{A}_w$. The results about pre-tangents can be easily deduced.

Lemma 3.15. *For $w \in W_*$, $\sigma \geq 0$ and $k \geq \lceil \sigma/2 \rceil$, $P_{\mathcal{H}_{k-1}}\mathcal{A}_w \in \mathcal{L}\big(H^\sigma(K), l^2(\mathcal{H}_{k-1}, A_w; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})\big)$.*

Proof. For $\sigma = 0$, it is easy to see the assertion by using Lemma 3.8. In fact, $\mathcal{H}_{k-1}(K)$ is a finite dimensional subspace of $C(K)$, so there is $g_k \in C(K \times K)$ such that $P_{\mathcal{H}_{k-1}}f(x) = \int_K g_k(x,y)f(y)d\mu(y)$. Then a similar argument as in the proof of Proposition 3.9 works.

For the $\sigma \geq 2$ case, the lemma is a consequence of Proposition 3.9, noticing that

$$P_{\mathcal{H}_{k-1}}A_w^{n+1}f - A_w P_{\mathcal{H}_{k-1}}A_w^n f = P_{\mathcal{H}_{k-1}}\big(A_w^{n+1}f - A_{w,\mathcal{H}_{k-1}}A_w^n f\big).$$

For $0 < \sigma < 2$, the assertion can be proved by using complex interpolation. □

Proposition 3.16. *For $\omega = \dot{w} \in \mathcal{P}$, $\sigma \geq 0$ and $k \in \mathbb{Z}^+$, there is a recovering map*

$$\mathcal{R}_{k,w} \in \mathcal{L}\big(l^2(\mathcal{H}_{k-1}, A_w; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}), H^\sigma(K)\big)$$

such that $P_{\mathcal{H}_{k-1}}\mathcal{A}_w\mathcal{R}_{k,w} = id$. In addition, $\mathcal{R}_{k,w}(\cdot)$ vanishes in a neighbourhood of $V_0 \setminus \{\pi(\omega)\}$.

Proof. First we assume that F_wK is bounded away from $V_0 \setminus \{\pi(\omega)\}$. Then, for each $h \in \mathcal{H}_{k-1}$, obviously there is $f \in dom\Delta^\infty$ such that $A_w f = h, P_{\mathcal{H}_{k-1}}f = 0$ and f vanishes in a neighbourhood of $V_0 \setminus \{\pi(\omega)\}$. By a standard argument, there is a linear map $R_{k,w} : \mathcal{H}_{k-1} \to dom\Delta^\infty$ such that $A_w R_{k,w}(h) = h$, $P_{\mathcal{H}_{k-1}}R_{k,w}(h) = 0$ and $R_{k,w}(h)$ vanishes in a neighbourhood of $V_0 \setminus \{\pi(\omega)\}$.

By a same reason, there is a map $R'_{k,w} : \mathcal{H}_{k-1} \to dom\Delta^\infty$ such that $A_w R'_{k,w}(h) = A_w h$, $P_{\mathcal{H}_{k-1}}R'_{k,w}(h) = h$ and $R'_{k,w}(h)$ vanishes in a neighbourhood of $V_0 \setminus \{\pi(\omega)\}$.

Now for any $\boldsymbol{h} = \{h_n\}_{n\geq 0} \in l^2(\mathcal{H}_{k-1}, A_w; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})$, we define

$$\mathcal{R}_{k,w}(\boldsymbol{h}) = R'_{k,w}(h_0) + \sum_{n=1}^{\infty} R_{k,w}(h_n - A_w h_{n-1}) \circ F_w^{-n+1}.$$

We need to show that $\mathcal{R}_{k,w}$ is well defined and is in $\mathcal{L}\big(l^2(\mathcal{H}_{k-1}, A_w; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}), H^\sigma(K)\big)$.

First, we consider $\sigma = 2k' < 2k$. Write $f_n = R_{k,w}(h_n - A_w h_{n-1}) \circ F_w^{-n+1}, n \geq 1$ and $f_0 = R'_{k,w}(h_0)$ for short. Then we see that

$$\begin{aligned}\|\{\Delta^{k'}f_n\}_{n\geq 0}\|_{l^2(L^\infty(K);\mu_w^{-1/2})} &\lesssim \|h_0\|_{\mathcal{H}_{k-1}} + \|\{h_n - A_w h_{n-1}\}_{n\geq 0}\|_{l^2(\mathcal{H}_{k-1};r_w^{k'}\mu_w^{k'-1/2})}\\ &= \|\boldsymbol{h}\|_{l^2(\mathcal{H}_{k-1},A_w;r_w^{k'}\mu_w^{k'-1/2})}.\end{aligned}$$

Write $Z = K \setminus F_wK$. Then we have

$$\begin{aligned}&\Big\|\sum_{m=0}^{\infty}|\Delta^{k'}f_m|\Big\|_{L^2(K)} = \Big\|\mu_w^{n/2}\|A_w^n\sum_{m=0}^{n+1}|\Delta^{k'}f_m|\|_{L^2(Z)}\Big\|_{l^2} \lesssim \Big\|\mu_w^{n/2}\sum_{m=0}^{n+1}\|\Delta^{k'}f_m\|_{L^\infty(K)}\Big\|_{l^2}\\ =&\Big\|\sum_{m=-1}^{\infty}1_{n\geq m}\mu_w^{m/2}\mu_w^{(n-m)/2}\|\Delta^{k'}f_{n-m}\|_{L^\infty(K)}\Big\|_{l^2} \lesssim \|\{\Delta^{k'}f_n\}_{n\geq 0}\|_{l^2(L^\infty(K);\mu_w^{-1/2})}.\end{aligned}$$

Thus, $\big\|\sum_{m=0}^\infty|\Delta^{k'}f_m|\big\|_{L^2(K)}\lesssim\|\boldsymbol{h}\|_{l^2(\mathcal{H}_{k-1},A_w;r_w^{k'}\mu_w^{k'-1/2})}$. By a same argument, we have

$$\big\|\sum_{m=0}^\infty|f_m|\big\|_{L^2(K)}\lesssim\|\boldsymbol{h}\|_{l^2(\mathcal{H}_{k-1},A_w;\mu_w^{-1/2})}\leq\|\boldsymbol{h}\|_{l^2(\mathcal{H}_{k-1},A_w;r_w^{k'}\mu_w^{k'-1/2})}. \tag{3.1}$$

The above estimates show that $\mathcal{R}_{k,w}(\boldsymbol{h})=\sum_{m=0}^\infty f_m$ is well defined in $H^{2k'}(K)$, and $\mathcal{R}_{k,w}\in\mathcal{L}\big(l^2(\mathcal{H}_{k-1},A_w;r_w^{k'}\mu_w^{k'-1/2}),H^{2k'}(K)\big)$.

Next, we consider $\sigma=2k'\geq 2k$. We can see that for any $n\geq1$, $\Delta^{k'}f_n$ is supported in $F_w^{n-1}Z$ with $\|\Delta^{k'}f_n\|_{L^2(K)}\lesssim r_w^{-k'n}\mu_w^{-(k'-1/2)n}\|h_n-A_wh_{n-1}\|_{\mathcal{H}_{k-1}}$. This shows that

$$\|\Delta^{k'}\mathcal{R}_{k,w}(\boldsymbol{h})\|_{L^2(K)}\lesssim\|\boldsymbol{h}\|_{l^2(\mathcal{H}_{k-1},A_w;r_w^{k'}\mu_w^{k'-1/2})}. \tag{3.2}$$

Combining estimates (3.1) and (3.2), we still see that $\mathcal{R}_{k,w}\in\mathcal{L}\big(l^2(\mathcal{H}_{k-1},A_w;r_w^{k'}\mu_w^{k'-1/2}),H^{2k'}(K)\big)$.

Using complex interpolation, we see that for any $\sigma\geq0$, we have

$$\mathcal{R}_{k,w}\in\mathcal{L}\big(l^2(\mathcal{H}_{k-1},A_w;r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}),H^\sigma(K)\big).$$

Lastly, it is easy to check that $P_{\mathcal{H}_{k-1}}\mathcal{A}_w\mathcal{R}_{k,w}=id$ from the definition.

For the case $F_wK\cap\big(V_0\setminus\{\pi(\omega)\}\big)\neq\emptyset$, we only need slightly modify the definition of $R_{k,w}$ and $R'_{k,w}$. □

Now we are able to provide the exact space of pre-tangents for $H^\sigma(K)$.

Theorem 3.17. *Let $\sigma\geq0$, $k\geq\lceil\sigma/2\rceil$ and $\omega=\tau\dot{w}\in\mathcal{P}$. Let l be the same integer as in Definition 3.4.*

(a). The k-pre-tangent mapping $\mathcal{T}_{\mathcal{H}_{k-1},\omega}$ is bounded and surjective from $H^\sigma(K)$ to $l^2(\mathcal{H}_{k-1},A_w;r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})$.

(b). For $\boldsymbol{h}\in\mathcal{H}_{k-1}^{\mathbb{Z}^+}$, write $\mathcal{R}_{\mathcal{H}_{k-1},\omega}(\boldsymbol{h})=\mathcal{R}_{k,w}(\boldsymbol{h})\circ F_w^{-l}F_\tau^{-1}$. Then $\mathcal{R}_{\mathcal{H}_{k-1},\omega}$ is bounded from $l^2(\mathcal{H}_{k-1},A_w;r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})$ to $H^\sigma(K)$, and $\mathcal{T}_{\mathcal{H}_{k-1},\omega}\circ\mathcal{R}_{\mathcal{H}_{k-1},\omega}=id$.

(c). Assume $f\in H^\sigma(K),\sigma>\frac{d_S}{2}$ and $\mathcal{T}_{\mathcal{H}_{k-1},\omega}f=\mathbf{0}$. Then $\mathcal{A}_w(A_\tau f)\in l^2(L^\infty(K);r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})$.

Proof. (a) and (b) are easy consequences of Lemma 3.15 and Proposition 3.16. (c) can be observed from Proposition 3.9 and Proposition 3.12, noticing that $A_{w,\mathcal{H}_{k-1}}A_w^n(A_\tau f)=0,n\geq l$ from the assumption of (c). □

4. The Sobolev spaces $H_0^\sigma(K)$

We now proceed to study the Sobolev spaces $H_0^\sigma(K)$. In this section, we will make a full characterization of the relationship between $H_0^\sigma(K)$ and $H^\sigma(K)$ in terms of the boundary behavior of functions. We assume both the conditions (**C1**) and (**C2**) in this section.

Recall that our domain is $\Omega=K\setminus V_0$ with boundary V_0. The space of smooth functions with compact support is defined as

$$\mathcal{D}(\Omega)=\{f\in C_c(\Omega):\Delta^kf\in C_c(\Omega),\forall k\geq0\},$$

where $C_c(\Omega)$ is the space of continuous functions with compact supports in Ω. See [36] for basic properties of $\mathcal{D}(\Omega)$ in the fractal settings. Naturally, $\mathcal{D}(\Omega)$ is a subspace of $H^\sigma(K)$, $\forall\sigma\geq0$.

Definition 4.1. *For $\sigma\geq0$, define $H_0^\sigma(K)$ as the closure of $\mathcal{D}(\Omega)$ in $H^\sigma(K)$ with respect to the norm $\|\cdot\|_{H^\sigma(K)}$.*

Theorem 4.2. *For $\sigma \geq 0$, we have $H_0^\sigma(K) = \{f \in H^\sigma(K) : T_\omega^{(\sigma)}(f) = 0, \forall \omega \in \mathcal{P}\}$. In particular, $H_0^\sigma(K) = H^\sigma(K)$ if $\sigma \leq \frac{d_S}{2}$.*

Both Theorem 3.14 and 4.2 have elegant analogues for domains $\Omega \in \mathbb{R}^n$ with good boundary. Related development can be found in [27] (Chapter 1, Section 9 and 10).

In the rest of this section, we will focus on the proof of Theorem 4.2. Since $T_\omega^{(\sigma)}$ is continuous by Theorem 3.14 and $T_\omega^{(\sigma)}|_{\mathcal{D}(\Omega)} = 0$, we can see that one direction of the theorem holds, i.e.,

$$H_0^\sigma(K) \subset \{f \in H^\sigma(K) : T_\omega^{(\sigma)}(f) = 0, \forall \omega \in \mathcal{P}\}.$$

For the other direction, we first develop two lemmas (Lemma 4.4 and Lemma 4.6).

The first lemma is an application of Theorem 3.17. Recall the definition of $\mathcal{T}_{\mathcal{H}_{k-1},\omega}$ and $\mathcal{R}_{\mathcal{H}_{k-1},\omega}$ in Definition 3.4 and Theorem 3.17.

Definition 4.3. *For $\sigma \geq 0$, $k \geq \lceil \sigma/2 \rceil$, define $ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}} = \{f \in H^\sigma(K) : \mathcal{T}_{\mathcal{H}_{k-1},\omega} f = \mathbf{0}, \forall \omega \in \mathcal{P}\}$, with induced norm $\|\cdot\|_{H^\sigma(K)}$.*

Lemma 4.4. *Let $\sigma \geq 0$, $k \geq \lceil \sigma/2 \rceil$.*

(a). For each $\omega = \tau \dot{w} \in \mathcal{P}$ and $\sigma \geq 0$, we have

$$\|\mathcal{R}_{\mathcal{H}_{k-1},\omega}(\boldsymbol{h})\|_{H^\sigma(K)} \asymp \|\boldsymbol{h}\|_{l^2(\mathcal{H}_{k-1},A_w;r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})}.$$

(b). $H^\sigma(K) = \ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}} \oplus \big(\oplus_{\omega\in\mathcal{P}} \mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(l^2(\mathcal{H}_{k-1}, A_w; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})\big)\big)$, $\forall \sigma \geq 0$.

(c). Let $\sigma' > \sigma \geq 0$, then $ker_{\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}}$ is a dense subspace of $ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}}$.

Proof. (a) is an easy consequence of Theorem 3.17 (a), (b).

(b). Given any function $f \in H^\sigma(K)$, we can easily see that

$$f - \sum_{\omega\in\mathcal{P}} \mathcal{R}_{\mathcal{H}_{k-1},\omega} \circ \mathcal{T}_{\mathcal{H}_{k-1},\omega}(f) \in \ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}}.$$

The decomposition of f is obviously unique.

(c). For a subspace $X \subset H^\sigma(K)$, we write $\overline{X}$ for the closure of X in $H^\sigma(K)$ for short. For convenience, we also write $X_{\omega,\sigma} = \mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(l^2(\mathcal{H}_{k-1}, A_w; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2}\big)$ for short.

It is obvious that $\ker_{\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}} \subset \ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}}$, and $X_{\omega,\sigma'} \subset X_{\omega,\sigma}$, which leads to

$$\overline{H^{\sigma'}(K)} = \overline{\ker_{\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}}} \oplus (\oplus_{\omega\in\mathcal{P}} \overline{X_{\omega,\sigma'}}).$$

However, we know that $\overline{H^{\sigma'}(K)} = H^\sigma(K)$ by a standard argument. This implies that $\overline{\ker_{\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}}} = \ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}}$. □

The second lemma will focus on smooth functions. We will use *smooth bump functions* developed by Rogers, Strichartz and Teplyaev in [35]. In particular, we need the following easy consequence (See Theorem 4.3 and estimate (4.7) in [35]).

Proposition 4.5. *Let $k \geq 1, p \in V_0$ and $f \in dom\Delta^\infty$. There is a function $g \in dom\Delta^\infty$ such that*

$$\begin{cases} \|g\|_{H^{2k}(K)} \lesssim \|f\|_{H^{2k}(K)}, \\ \Delta^j g(q) = \Delta^j f(q), \partial_n \Delta^j g(q) = \partial_n \Delta^j f(q), \forall j \geq 0, \forall q \in V_0 \setminus \{p\}, \end{cases}$$

and the support of g is away from p.

Using Proposition 4.5, we have the following lemma.

Lemma 4.6. *Let $k\geq 1$ and $f\in H^{2k}(K)$. If $\forall\omega=\tau\dot{w}\in\mathcal{P}$, $\mathcal{T}_{\mathcal{H}_{k-1},\omega}f\in l^2(\mathcal{H}_{k-1};r_w^k\mu_w^{k-1/2})$, then $f\in H_0^{2k}(K)$.*

Proof. For each $\omega=\tau\dot{w}\in\mathcal{P}$, by an easy estimate, we can see that

$$\|A_w^nA_\tau f-P_{\mathcal{H}_{k-1}}A_w^nA_\tau f\|_{H^{2k}(K)}\lesssim r_w^{kn}\mu_w^{kn}\|A_w^nA_\tau\Delta^kf\|_{L^2(K)}=o(r_w^{kn}\mu_w^{(k-1/2)n}).$$

In addition, it is obvious from the assumption that

$$\|P_{\mathcal{H}_{k-1}}A_w^nA_\tau f\|_{H^{2k}(K)}\asymp\|P_{\mathcal{H}_{k-1}}A_w^nA_\tau f\|_{\mathcal{H}_{k-1}(K)}=o(r_w^{kn}\mu_w^{(k-1/2)n}).$$

Thus we have $\|A_w^nA_\tau f\|_{H^{2k}(K)}=o(r_w^{kn}\mu_w^{(k-1/2)n})$.

Now, we construct $g\in\mathcal{D}(\Omega)$ that well approximates f in $H^{2k}(K)$. For any $\varepsilon>0$, we can do the following.

1. Choose a large n such that $\|A_w^nA_\tau f\|_{H^{2k}(K)}\leq\varepsilon r_w^{kn}\mu_w^{(k-1/2)n}r_\tau^k\mu_\tau^{k-1/2},\forall\omega\in\mathcal{P}$.

2. Let P_t be the heat kernel generated by the Dirichlet Laplacian Δ_D. Choose t small enough so that $\|f-P_tf\|_{H^{2k}(K)}\leq\varepsilon$ and $\|A_w^nA_\tau(f-P_tf)\|_{H^{2k}(K)}\leq\varepsilon r_w^{kn}\mu_w^{(k-1/2)n}r_\tau^k\mu_\tau^{k-1/2}$.

3. By Proposition 4.5, for each ω, we can find a g_ω supported in $F_w^nF_\tau\setminus\pi(\omega)$ such that

$$\begin{aligned}\|\Delta^kg_\omega\|_{L^2(F_w^nF_\tau K)}&\leq r_w^{-kn}\mu_w^{-(k-1/2)n}r_\tau^{-k}\mu_\tau^{-k+1/2}\|A_w^nA_\tau g_\omega\|_{H^{2k}(K)}\\&\leq Cr_w^{-kn}\mu_w^{-(k-1/2)n}r_\tau^{-k}\mu_\tau^{-k+1/2}\|A_w^nA_\tau P_tf\|_{H^{2k}(K)}\leq 2C\varepsilon,\end{aligned}$$

and $\Delta^jA_w^nA_\tau g_\omega(q)=\Delta^jA_w^nA_\tau P_tf(q),\partial_n\Delta^jA_w^nA_\tau g_\omega(q)=\partial_n\Delta^jA_w^nA_\tau P_tf(q),\forall j\geq 0,q\in V_0\setminus\{\pi(\omega)\}$. Replace $P_tf|_{F_w^nF_\tau(K)}$ with g_ω for each ω, and name the induced function g. Clearly, $g\in\mathcal{D}(\Omega)$. One can then check that

$$\|f-g\|_{H^{2k}(K)}\leq C'\|\Delta^kf-\Delta^kg\|_{L^2(K)}\leq C'(1+2\#\mathcal{P}+2C\#\mathcal{P})\varepsilon,$$

noticing that both f,g are in $H_D^{2k}(K)$. The lemma is proved by choosing ε arbitrarily. □

Proof of Theorem 4.2. It suffices to show $\{f\in H^\sigma(K):T_\omega^{(\sigma)}(f)=0,\forall\omega\in\mathcal{P}\}\subset H_0^\sigma(K)$. Choose k large enough so that $\sigma\leq 2k$. By Lemma 4.6, we have

$$\ker_{2k}\mathcal{T}_{\mathcal{H}_{k-1}}\oplus\Big(\oplus_{\omega\in\mathcal{P}}\mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(l^2(\mathcal{H}_{k-1};r_w^k\mu_w^{k-1/2})\big)\Big)\subset H_0^{2k}(K)\subset H_0^\sigma(K).$$

As a consequence, by using Lemma 4.4 (a) and (c), we get

$$\ker_\sigma\mathcal{T}_{\mathcal{H}_{k-1}}\oplus\Big(\oplus_{\omega\in\mathcal{P}}\mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(\overline{l^2(\mathcal{H}_{k-1};r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})}\big)\Big)\subset H_0^\sigma(K).$$

On the other hand, by Theorem 3.17 (b),(c) and Lemma 4.4 (b), and using Lemma 3.6, we can see that

$$\{f\in H^\sigma(K):T_\omega^{(\sigma)}(f)=0,\forall\omega\in\mathcal{P}\}=\ker_\sigma\mathcal{T}_{\mathcal{H}_{k-1}}\oplus\Big(\oplus_{\omega\in\mathcal{P}}\mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(\overline{l^2(\mathcal{H}_{k-1};r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})}\big)\Big).$$

The assertion follows immediately. □

As a consequence of Theorem 4.2, we have the following characterization of $H_0^\sigma(K)$.

Theorem 4.7. *Let $k\geq 1$ be an integer, and let $0\leq\sigma\leq 2k$. Then we have*

$$H_0^\sigma(K)=\ker_\sigma\mathcal{T}_{\mathcal{H}_{k-1}}\oplus\Big(\oplus_{\omega\in\mathcal{P}}\mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(\overline{l^2(\mathcal{H}_{k-1};r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})}\big)\Big).\tag{4.1}$$

5. Interpolation of $H^\sigma(K)$: $\sigma \geq 0$

Now we are ready to turn to the second topic, the interpolation theorems. In this section, we prove some interpolation theorems for Sobolev spaces with non-negative orders, and we will combine these results in a final theorem on Sobolev spaces with real orders in Section 6. We assume both (**C1**) and (**C2**) in this section.

Lemma 5.1. *Let (Z_0, Z_1) be an interpolation couple, which means Z_0 and Z_1 are continuously embedded in a same Hausdorff topological vector space. Let $Z_0 = X_0 \oplus Y_0$ and $Z_1 = X_1 \oplus Y_1$, and assume that $(X_0 + X_1) \cap (Y_0 + Y_1) = \{0\}$. Then we have*
(a). $[Z_0, Z_1]_\theta = [X_0, X_1]_\theta \oplus [Y_0, Y_1]_\theta$;
(b). Assume that $[Z_0, Z_1]_\theta = \tilde{X} \oplus \tilde{Y}$ and $\tilde{X} \subset X_0 + X_1, \tilde{Y} \subset Y_0 + Y_1$, then $\tilde{X} = [X_0, X_1]_\theta$ and $\tilde{Y} = [Y_0, Y_1]_\theta$, with equivalent norms.

Proof. (a). Since $Z_0+Z_1 = (X_0+X_1)+(Y_0+Y_1)$ with $(X_0+X_1)\cap(Y_0+Y_1) = \{0\}$, we can define the natural projection $P : Z_0 + Z_1 \to X_0 + X_1$ such that $(1 - P) : Z_0 + Z_1 \to Y_0 + Y_1$. It is easy to check that $P \in \mathcal{L}([Z_0, Z_1]_\theta, [X_0, X_1]_\theta)$ and $1-P \in \mathcal{L}([Z_0, Z_1]_\theta, [Y_0, Y_1]_\theta)$ by using complex interpolation. So $[Z_0, Z_1]_\theta = [X_0, X_1]_\theta + [Y_0, Y_1]_\theta$ with $[X_0, X_1]_\theta \cap [Y_0, Y_1]_\theta = \{0\}$.

It remains to check that $\|x\|_{[Z_0,Z_1]_\theta} \asymp \|x\|_{[X_0,X_1]_\theta}$ for any $x \in [X_0, X_1]_\theta$ and $\|y\|_{[Z_0,Z_1]_\theta} \asymp \|y\|_{[Y_0,Y_1]_\theta}$ for any $y \in [Y_0, Y_1]_\theta$. Let i_X be the embedding map from $X_0 + X_1 \to Z_0 + Z_1$. Then, by using complex interpolation, one can see that $i_X \in \mathcal{L}([X_0, X_1]_\theta, [Z_0, Z_1]_\theta)$. As a consequence, $\|x\|_{[X_0,X_1]_\theta} = \|Px\|_{[X_0,X_1]_\theta} \lesssim \|x\|_{[Z_0,Z_1]_\theta} = \|i_X x\|_{[Z_0,Z_1]_\theta} \lesssim \|x\|_{[X_0,X_1]_\theta}$, where we also use the fact that $P \in \mathcal{L}([Z_0, Z_1]_\theta, [X_0, X_1]_\theta)$. The proof for $[Y_0, Y_1]_\theta$ is the same.

(b). Clearly $[X_0, X_1]_\theta = P([Z_0, Z_1]_\theta) = \tilde{X}$, and $[Y_0, Y_1]_\theta = (1 - P)([Z_0, Z_1]_\theta) = \tilde{Y}$. The estimate of the norms is obvious. □

Lemma 5.2. $\ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}}, \sigma \geq 0$ *(defined in Definition 4.3) is stable under complex interpolation, i.e., for $2k \geq \sigma > \sigma' \geq 0$, it holds that*

$$[\ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}}, \ker_{\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}}]_\theta = \ker_{(1-\theta)\sigma+\theta\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}}, \forall\theta \in [0, 1].$$

Proof. We need to use the fact that $H^\sigma_D(K)$ is stable under complex interpolation, see [38]. In other words, $H_D^{(1-\theta)\sigma+\theta\sigma'}(K) = [H^\sigma_D(K), H^{\sigma'}_D(K)]_\theta, \forall\theta \in [0, 1]$. It is easy to see that $H^\sigma_D(K) = \ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}} \oplus X_\sigma$, where we write $X_\sigma = \oplus_{\omega\in\mathcal{P}} \mathcal{R}_{\mathcal{H}_{k-1},\omega}\mathcal{T}_{\mathcal{H}_{k-1},\omega}(H^\sigma_D(K))$ for convenience. Also, one can see $\big(\ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}} + \ker_{\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}}\big) \cap (X_\sigma + X_{\sigma'}) = \ker_{\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}} \cap X_{\sigma'} = \{0\}$, $\ker_{(1-\theta)\sigma+\theta\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}} \subset \ker_{\sigma'} \mathcal{T}_{\mathcal{H}_{k-1}}$ and $X_{(1-\theta)\sigma+\theta\sigma'} \subset X_{\sigma'}$. The lemma follows from Lemma 5.1 (b). □

Lemma 5.3. *Let X be a Banach space and A be a compact operator in $\mathcal{L}(X, X)$. Denote $\overline{l^2(X; \alpha)}$ the closure of $l^2(X; \alpha)$ in $l^2(X, A; \alpha)$. Then, we have*

$$[\overline{l^2(X; \alpha)}, \overline{l^2(X; \beta)}]_\theta = l^2(X; \alpha^{(1-\theta)}\beta^\theta),$$

for any $\infty > \alpha > \beta > 0$ and $0 < \theta < 1$.

Proof. Let's first make some observations of the special cases.

Claim 1: Let $\beta = |\lambda_l|$, then $[\overline{l^2(E_l; \alpha)}, \overline{l^2(E_l; \beta)}]_\theta = l^2(E_l; \alpha^{(1-\theta)}\beta^\theta)$.

In this case, by using Lemma 3.6, we see that

$$\begin{aligned}[\overline{l^2(E_l; \alpha)}, \overline{l^2(E_l; \beta)}]_\theta &= [l^2(E_l; \alpha), l^2(E_l, A; \beta)]_\theta = [l^2(E_l, A; \alpha), l^2(E_l, A; \beta)]_\theta \\ &= l^2(E_l, A; \alpha^{(1-\theta)}\beta^\theta) = l^2(E_l; \alpha^{(1-\theta)}\beta^\theta),\end{aligned}$$

where the last equality holds since $\alpha^{(1-\theta)}\beta^\theta > |\lambda_l|$.

Claim 2: Let $\alpha = |\lambda_l|$, then $[\overline{l^2(E_l;\alpha)}, \overline{l^2(E_l;\beta)}]_\theta = l^2(E_l;\alpha^{(1-\theta)}\beta^\theta)$.

First, choose $\theta_1, \theta_2 \in (0,1)$ such that $\theta_1 + \theta_2 - \theta_1\theta_2 = \theta$, we can see

$$\begin{aligned}
[\overline{l^2(E_l;\alpha)}, \overline{l^2(E_l;\beta)}]_\theta &= [l^2(E_l, A;\alpha), l^2(E_l;\beta)]_\theta \\
&= \big[[l^2(E_l, A;\alpha), l^2(E_l;\beta)]_{\theta_1}, l^2(E_l;\beta)\big]_{\theta_2} \\
&\subset [l^2(E_l, A;\alpha^{(1-\theta_1)}\beta^{\theta_1}), l^2(E_l;\beta)]_{\theta_2} \\
&= [l^2(E_l;\alpha^{(1-\theta_1)}\beta^{\theta_1}) \oplus \mathcal{S}_A(E_l), l^2(E_l;\beta) \oplus \{0\}]_{\theta_2} = l^2(E_l;\alpha^{(1-\theta)}\beta^\theta),
\end{aligned}$$

where we use Lemma 3.6 in the first and fourth lines, use the fact that $l^2(E_l;\beta) \subset l^2(E_l, A;\beta)$ in the third line, and use Lemma 5.1 in the last equality. On the other hand, we also have

$$l^2(E_l;\alpha^{1-\theta}\beta^\theta) = [l^2(E_l;\alpha), l^2(E_l;\beta)]_\theta \subset [\overline{l^2(E_l;\alpha)}, \overline{l^2(E_l;\beta)}]_\theta.$$

Combining the above two embedding relationships, noticing that both of them are continuous, we get the claim.

Now we return to prove the lemma. We need to consider four different cases, based on whether $\alpha, \beta \in \{|\lambda_l|\}_{l=0}^\infty$.

For the extreme case $\alpha = |\lambda_l|, \beta = |\lambda_{l'}|$ for some $l' > l$, we devide the space X into three pieces $X = \hat{E}_l \oplus \hat{E}_{l'} \oplus Y$, such that $\sigma(A, Y) = \sigma(A, X) \setminus |\{z : |z| = |\lambda_l| \text{ or } |\lambda_{l'}|\}$. By using Lemma 3.6, we see that

$$\begin{cases}
\overline{l^2(X;\alpha)} = \overline{l^2(\hat{E}_l;\alpha)} \oplus \overline{l^2(\hat{E}_{l'};\alpha)} \oplus l^2(Y;\alpha), \\
\overline{l^2(X;\beta)} = \overline{l^2(\hat{E}_l;\beta)} \oplus \overline{l^2(\hat{E}_{l'};\beta)} \oplus l^2(Y;\beta).
\end{cases}$$

The assertion then follows by using Lemma 5.1 (a) and the two claims. The other three cases can be proved similarly. □

The following are main theorems of this section.

Theorem 5.4. *The Sobolev spaces $H^\sigma(K)$ are stable under complex interpolation.*

Proof. For $\sigma > \sigma' \geq 0$, we can easily see that

$$[H^\sigma(K), H^{\sigma'}(K)]_\theta = H^{(1-\theta)\sigma+\theta'\sigma}(K), \forall \theta \in [0,1]$$

by using the Lemma 4.4 (b), Lemma 5.1 (a), Lemma 5.2 and the fact that $l^2(X, A;\alpha)$ is stable under complex interpolation. □

The interpolation result for $H_0^\sigma(K)$ is somewhat complicated. The result will coincide with the $\mathbb{R}^n$ case. Interested readers may read [27] for an interpolation theorem for $H_0^\sigma(\Omega)$ with $\Omega \subset \mathbb{R}^n$.

Definition 5.5. *For $\sigma \geq 0$, define*

$$H_{00}^\sigma(K) = \big\{f \in H^\sigma(K) : f \cdot \rho^{-\sigma} \in L^2(K)\big\},$$

where $\rho(x) = R(x, V_0)^{1/2+d_H/2}$ on K with $R(\cdot,\cdot)$ being the effective resistance metric. For each $f \in H_{00}^\sigma(K)$, we assign the norm

$$\|f\|_{H_{00}^\sigma(K)} = \|f\|_{H^\sigma(K)} + \|f\rho^{-\sigma}\|_{L^2(K)}.$$

Remark 1. $H^\sigma_{00}(K)$ are natural analogs of the *Lions-Magenes spaces*, see [27].
Remark 2. One can easily see that for all $2k \geq \sigma$ with $k \in \mathbb{N}$, we have

$$H^\sigma_{00}(K) = \ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}} \oplus \big(\oplus_{\omega\in\mathcal{P}} \mathcal{R}_{\mathcal{H}_{k-1},\omega}\big(l^2(\mathcal{H}_{k-1}; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})\big)\big).$$

In addition, for $f = f_0 + \sum_{\omega\in\mathcal{P}} \mathcal{R}_{\mathcal{H}_{k-1},\omega}(\boldsymbol{h}_\omega) \in H^\sigma_{00}(K)$ with $f_0 \in \ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}}$ and $\boldsymbol{h}_\omega \in l^2(\mathcal{H}_{k-1}; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})$, we have

$$\|f\|_{H^\sigma_{00}(K)} \asymp \|f_0\|_{H^\sigma(K)} + \sum_{\omega\in\mathcal{P}} \|\boldsymbol{h}_\omega\|_{l^2(\mathcal{H}_{k-1}; r_w^{\sigma/2}\mu_w^{(\sigma-1)/2})}.$$

We have the following interpolation theorems.

Theorem 5.6. *The spaces $H^\sigma_{00}(K)$ are stable under complex interpolation.*

Proof. The theorem is a consequence of the above Remark 2, by a same method as Theorem 5.4. □

Theorem 5.7. *Let $\sigma > \sigma' \geq 0$, then $[H^\sigma_0(K), H^{\sigma'}_0(K)]_\theta = H^{(1-\theta)\sigma+\theta\sigma'}_{00}(K), \forall\theta \in (0,1)$. In particular, $[H^\sigma_0(K), H^{\sigma'}_0(K)]_\theta = H^{(1-\theta)\sigma+\theta\sigma'}_0(K)$ if and only if $r_w^{\sigma_\theta/2}\mu_w^{(\sigma_\theta-1)/2} \notin \{|\lambda_{l,w}|\}_{l=0}^\infty$ for any $\omega = \tau\dot{w} \in \mathcal{P}$, with $\sigma_\theta = (1-\theta)\sigma + \theta\sigma'$.*

Proof. The first assertion follows from Theorem 4.7, Lemma 5.1 and 5.3. The second assertion is a consequence of Lemma 3.6. □

6. Interpolation of $H^\sigma(K)$: $\sigma \in \mathbb{R}$

In this section, we will extend the definition of Sobolev spaces $H^\sigma(K)$ to negative orders, and study the associated interpolation theorem. Readers may read Lions and Magenes's monograph [27] for classical theorems on bounded domains in $\mathbb{R}^n$. Part of the idea in this section is inspired by [27]. We will apply the theorems in Section 4 and 5, which are proven with conditions (**C1**) and (**C2**).

Definition 6.1. *For $\sigma \geq 0$, we define $H^{-\sigma}(K) = \big(H^\sigma_0(K)\big)'$, with the identification $H^0(K) = (H^0(K))' \subset H^{-\sigma}(K)$, noticing that $H^0(K) = H^0_0(K) = L^2(K)$.*

Remark 1. In the above definition, we naturally embed the space $H^0(K)$ into $H^{-\sigma}(K)$. More concretely, for each $f \in H^0(K)$, we can correspond it with a linear functional $\varphi_f \in \big(H^\sigma_0(K)\big)'$ by the formula $\varphi_f(g) = \int_K \overline{f(x)}g(x)d\mu(x) =< g, f >_{L^2(K)}, \forall g \in H^\sigma_0(K)$. As a consequence, we always have

$$H^\sigma(K) \subset H^{\sigma'}(K), \quad \forall \infty > \sigma > \sigma' > -\infty.$$

Remark 2. We also embed $H^0(K)$ into $\big(H^\sigma_{00}(K)\big)'$ in a same way. Notice that $H^{-\sigma}(K) = \big(H^\sigma_{00}(K)\big)'$ if and only if $r_w^{\sigma/2}\mu_w^{(\sigma-1)/2} \notin \{|\lambda_{l,w}|\}_{l\geq 0}$ for any $\omega = \tau\dot{w} \in \mathcal{P}$.

The following interpolation theorem is the main result in this section, which is a perfect analogue to the classical theorem.

Theorem 6.2. *Let $-\infty < \sigma' < \sigma < +\infty$, $0 < \theta < 1$ and $\sigma_\theta = (1-\theta)\sigma + \theta\sigma'$. We have*

$$[H^\sigma(K), H^{\sigma'}(K)]_\theta = \begin{cases} H^{\sigma_\theta}(K), & \text{if } \sigma_\theta \geq 0, \\ \big(H^{-\sigma_\theta}_{00}(K)\big)', & \text{if } \sigma_\theta < 0. \end{cases}$$

In particular,

$$[H^{\sigma}(K), H^{\sigma'}(K)]_\theta = H^{\sigma_\theta}(K)$$

if and only if $r_w^{-\sigma_\theta/2}\mu_w^{-(\sigma_\theta+1)/2} \notin \{|\lambda_{l,w}|\}_{l\geq 0}$ *for any* $\omega = \tau\dot{w} \in \mathcal{P}$.

In the rest of this section, we devote to prove Theorem 6.2.

6.1. **Lemmas.** We collect some lemmas first. For convenience, we let Z be a Hilbert space with inner product $< \cdot, \cdot >_Z$. Let $Z_1 \subset Z$ be a Banach space which is dense and continuously embedded in Z.

Define Z_{-1} as the dual space of Z_1, and embed Z into Z_{-1} by

$$z \to \varphi_z(\cdot) =< \cdot, z >_Z \in Z_{-1}. \tag{6.1}$$

By this, we have the relation $Z_1 \subset Z \subset Z_{-1}$. In addition, we have the following lemma due to Lions and Magenes [27] (Proposition 2.1).

Lemma 6.3. $[Z_{-1}, Z_1]_{1/2} = Z$.

The next lemma will play a key role in the proof of Theorem 6.2.

Lemma 6.4. *Let* $Z_1^{(0)} \subset Z_1$ *be a closed subspace of* Z_1 *and suppose* $Z_1^{(0)}$ *is dense in* Z*. Define* $Z_{-1}^{(0)}$ *in a same way as* Z_{-1}*. If there is a map* $L \in \mathcal{L}(Z,Z) \cap \mathcal{L}(Z_1, Z_1)$ *such that*

$$\begin{cases} L + id \in \mathcal{L}(Z,Z) \cap \mathcal{L}(Z_1, Z_1^{(0)}), \\ L^* - id \in \mathcal{L}(Z,Z) \cap \mathcal{L}(Z_1, Z_1^{(0)}), \end{cases}$$

where id *is the identity map and* L^* *is the adjoint operator of* L *with respect to* Z*, then*

$$[Z_{-1}^{(0)}, Z_1]_{1/2} = Z.$$

Proof. Let $\tilde{Z} = Z \times Z$ with inner product $< \tilde{z}, \tilde{z}' >_{\tilde{Z}} =< z_1, z_1' >_Z + < z_2, z_2' >_Z$, where $\tilde{z} = (z_1, z_2)$ and $\tilde{z}' = (z_1', z_2')$. Define $\tilde{Z}_1 = \{(z_1, z_2) \in Z_1 \times Z_1 : z_1 + z_2 = Z_1^{(0)}\}$ with norm $\|(z_1, z_2)\|_{\tilde{Z}_1} = \|z_1\|_{Z_1} + \|z_2\|_{Z_1}$. Like we have done before, we define $\tilde{Z}_{-1}$ with $\tilde{Z}$ naturally embedded in $\tilde{Z}_{-1}$, noticing here that $\tilde{Z}_1$ is dense in $\tilde{Z}$ by the assumption on $Z_1^{(0)}$.

Now we define the extension map $E \in \mathcal{L}(Z, \tilde{Z}) \cap \mathcal{L}(Z_1, \tilde{Z}_1)$ as follows

$$E(z) = (z, Lz), \quad \forall z \in Z.$$

The map E can be naturally extended to be $E \in \mathcal{L}(Z_{-1}^{(0)}, \tilde{Z}_{-1})$ with the formula

$$E\varphi(\tilde{z}) = \varphi(E^*\tilde{z}),$$

for any $\varphi \in Z_{-1}^{(0)}$ and $\tilde{z} \in \tilde{Z}_1$, noticing that $E^*\tilde{z} = E^*(z_1, z_2) = z_1 + L^*z_2 = z_1 + z_2 + (L^* - id)z_2 \in Z_1^{(0)}$. Therefore, we get $E \in \mathcal{L}(Z_{-1}^{(0)}, \tilde{Z}_{-1}) \cap \mathcal{L}(Z_1, \tilde{Z}_1)$. As a consequence, $E \in \mathcal{L}([Z_{-1}^{(0)}, Z_1]_{1/2}, \tilde{Z})$ by using complex interpolation and Lemma 6.3.

We also define a restriction map $R : \tilde{Z}_{-1} \to Z_{-1}^{(0)}$ by the formula

$$(R\tilde{\varphi})(z) = \tilde{\varphi}(z, 0), \quad \forall \tilde{\varphi} \in \tilde{Z}_{-1} \text{ and } z \in Z_1^{(0)}.$$

It is then easy to see that RE is the identity map from $Z^{(0)}_{-1}$ to $Z^{(0)}_{-1}$. In addition, we have $R(\tilde{Z})=Z$. Thus we get

$$[Z^{(0)}_{-1},Z_1]_{1/2}=RE([Z^{(0)}_{-1},Z_1]_{1/2})\subset R(\tilde{Z})=Z.$$

On the other hand, we have $Z=[Z^{(0)}_{-1},Z^{(0)}_1]_{1/2}\subset[Z^{(0)}_{-1},Z_1]_{1/2}$ by using Lemma 6.3. This finishes the proof. □

6.2. **A decomposition by projection.** Lemma 6.4 provides the strategy of the proof. Nevertheless, we need to overcome the difficulty that multiplication does not preserve smoothness in the fractal case [6]. In this part, for $k\in\mathbb{N}$, we focus on constructing a subspace $S_{k,w}$ in $L^2(K)$, such that the projections of functions in $H^{2k}(K)$ on $S_{k,w}$ maintain the smoothness, which will give us a new decomposition of the space $H^{2k}(K)$. This will play the role of multiplication by smooth bump functions.

Lemma 6.5. *For $\omega=\dot{w}\in\mathcal{P}$, assuming $F_wK\cap V_0=\{\pi(\omega)\}$ without loss of generality, there is a linear map $\tilde{R}_{k,w}:\mathcal{H}_{k-1}\to dom\Delta^\infty$ such that $A_w\tilde{R}_{k,w}(h)=A_wh$, $P_{\tilde{R}_{k,w}(\mathcal{H}_{k-1})}(h)=\tilde{R}_{k,w}(h)$ and $\tilde{R}_{k,w}(h)$ is supported away from $V_0\setminus\{\pi(\omega)\}$.*

Proof. To achieve this, we choose a basis $\{h_1,h_2,\cdots,h_n\}$ of $\mathcal{H}_{k-1}$, and denote

$$a_{ij}=<h_i,h_j>_{L^2(F_wK)},\quad 1\leq i,j\leq n.$$

It is clear that we can find $\tilde{h}_i\in dom\Delta^\infty,1\leq i\leq n$, such that $A_w\tilde{h}_i=A_wh_i$, the support of $\tilde{h}_i$ is a small neighbourhood of F_wK, and $<\tilde{h}_i,h_j>_{L^2(K)}=a_{ij}$. In addition, we can assume that

$$<\tilde{h}_i,\tilde{h}_j>_{L^2(K)}=\varepsilon_{ij}+a_{ij},\quad 1\leq i,j\leq n$$

with $\varepsilon=\max_{i,j}\{|\varepsilon_{ij}|\}$ small enough so that we can find $f_i\in dom\Delta^\infty$ supported in some compact subsets of $K\setminus F_wK$ away from the boundary, satisfying

$$\begin{cases}<f_i,\tilde{h}_j>_{L^2(K)}=0, & \forall 1\leq i,j\leq n,\\ <f_i,f_j>_{L^2(K)}=\delta_{ij}\varepsilon, & \forall 1\leq i,j\leq n,\\ <f_i,h_j>_{L^2(K)}=\varepsilon_{ij}+\delta_{ij}\varepsilon, & \forall 1\leq i,j\leq n.\end{cases}$$

Set $\tilde{R}_{k,w}(h_i)=\tilde{h}_i+f_i$, and extend $\tilde{R}_{k,w}$ to be the linear map $\mathcal{H}_{k-1}\to dom\Delta^\infty$. One can then check that

$$<h_i,\tilde{R}_{k,w}(h_j)>_{L^2(K)}=<\tilde{R}_{k,w}(h_i),\tilde{R}_{k,w}(h_j)>_{L^2(K)}=a_{ij}+\varepsilon_{ij}+\delta_{ij}\varepsilon,$$

and thus $P_{\tilde{R}_{k,w}(\mathcal{H}_{k-1})}(h_i)=\tilde{R}_{k,w}(h_i)$ for any $1\leq i\leq n$. The lemma follows immediately. □

Definition 6.6. *Let $\omega=\dot{w}\in\mathcal{P}$, $k\geq1$ be an integer.*

(a). Write $f_{w,h}$ for $\tilde{R}_{k,w}(h)$ for short. We omit k since $\tilde{R}_{k,w}$ can be defined consistently for different k's.

(b). Let $S_{k,w}$ be the subspace of $L^2(K)$ spanned by the functions $\{f_{w,h}\circ F_w^{-n}:h\in\mathcal{H}_{k-1},n\geq0\}$.

We have the following theorem.

Theorem 6.7. *Let $k \geq 1$ be an integer, $f \in H^{2k}(K)$, and $\omega = \tau\dot{w} \in \mathcal{P}$.*

(a). If $A_\tau f \perp S_{k,w}$ in $L^2(K)$, we have $T^{(2k)}_\omega f = 0$.

(b). $P_{S_{k,w}} A_\tau f \in H^{2k}(K)$, and $f - \mu_\tau A^*_\tau P_{S_{k,w}} A_\tau f \in H^{2k}(K)$, where A^*_τ is the adjoint operator of A_τ in $L^2(K)$, which can be expressed by $A^*_\tau g = \mu_\tau^{-1} g \circ F_\tau^{-1}, \forall g \in L^2(K)$.

Proof. (a). Let $h = T^{(2k)}_\omega f$ and $\tilde{f} = A_\tau f - h$. Then we have

$$\begin{aligned} < f_{w,A^n_w h} \circ F_w^{-n} \circ F_\tau^{-1}, f >_{L^2(K)} &= \mu^n_w \mu_\tau < f_{w,A^n_w h}, A^n_w A_\tau f >_{L^2(K)} \\ &= \mu^n_w \mu_\tau \big(< f_{w,A^n_w h}, A^n_w h >_{L^2(K)} + < f_{w,A^n_w h}, A^n_w \tilde{f} >_{L^2(K)} \big) \\ &= \mu^n_w \mu_\tau \big(\|f_{w,A^n_w h}\|^2_{L^2(K)} + o(\lambda^n_{l_w(2k),\omega}) \|f_{w,A^n_w h}\|_{L^2(K)} \big). \end{aligned}$$

Thus the left side equals 0 for any $n \geq 0$ only if $T^{(2k)}_\omega f = h = 0$.

(b). Without loss of generality, we consider the case that $\omega = \dot{w}$. For each $f \in H^{2k}(K)$, we will construct a series $\sum_{n=0}^\infty f^{(n)}$ converging in $H^{2k}(K)$, where each $f^{(n)}$ takes the form $f^{(n)} = f_{w,h} \circ F_w^{-n}$ for some $h \in \mathcal{H}_{k-1}$, so that $P_{S_{k,w}} f = \sum_{n=0}^\infty f^{(n)}$.

First, we look at some special functions. Let $f \in L^2(K)$ such that $A^l_w f \in \mathcal{H}_{k-1}$ for some $l \geq 0$. Denote $S^{(l)}_{k,w} = \{f_{w,h} \circ F_w^{-n} : h \in \mathcal{H}_{k-1}, 0 \leq n \leq l\}$, and write $P_{S^{(l)}_{k,w}} f = \sum_{n=0}^l f^{(n)}$. Clearly, $g = f - \sum_{n=0}^{l-1} f^{(n)}$ is k-multiharmonic in $F^l_w K$, and so $f^{(l)} = f_{w,A^l_w g} \circ F_w^{-l}$ by Lemma 6.5. As a consequence, we have $f - P_{S^{(l)}_{k,w}} f = 0$ on $F^{l+1}_w K$, which shows that $f - P_{S^{(l)}_{k,w}} f \perp S_{k,w}$. By this observation, we have the following construction.

Step 1: For any $f \in L^2(K)$ such that $A^l_w f \in \mathcal{H}_{k-1}$ for some $l \geq 0$, we can write $P_{S_{k,w}} f = \sum_{n=0}^l f^{(n)}$, where each $f^{(n)}$ takes the form $f^{(n)} = f_{w,h} \circ F_w^{-n}$ for some $h \in \mathcal{H}_{k-1}$.

Step 2: For any $f \in L^2(K)$ such that $A^l_w f \in \mathcal{H}_{k-1}$ for some $l \geq 0$, we have by induction

$$\|f^{(n)}\|_{L^\infty(K)} \lesssim \|\sum_{m=0}^n f^{(m)}\|_{L^2(F^n_w K \setminus F^{n+1}_w K)} + \sum_{m=0}^{n-1} \|f^{(m)}\|_{L^\infty(K)} \lesssim 2^n \|f\|_{L^2(K)}, \text{ for } n \geq 0.$$

Therefore we can continuously extend the definition of $f^{(n)}, n \geq 0$ to general functions f in $L^2(K)$.

We have some observations on the sequence $\{f^{(n)}\}_{n \geq 0}$.

Observation 1: For any $f \in L^2(K)$ and $n \geq 1$, $A^{n-1}_w f^{(n)} = (A^{n-1}_w f)^{(1)}$.

Proof of Observation 1. We only need to consider the case that $A^l_w f \in \mathcal{H}_{k-1}$ for some $l \geq 0$. Let $g = f - \sum_{m=0}^{n-2} f^{(m)}$. Then we have

$$A^{n-1}_w P_{S_{k,w}} g = A^{n-1}_w P_{S^{(n-1)+}_{k,w}} g = P_{S_{k,w}} \big(A^{n-1}_w g \big),$$

where $S^{(n-1)+}_{k,w} = \{f_{w,h} \circ F_w^{-m} : h \in \mathcal{H}_{k-1}, m \geq n-1\}$. So we have $A^{n-1}_w f^{(n)} = A^{n-1}_w g^{(n)} = (A^{n-1}_w g)^{(1)}$. On the other hand, we have $(A^{n-1}_w f)^{(1)} = (A^{n-1}_w g)^{(1)}$ as $A^{n-1}_w (f - g) \in \mathcal{H}_{k-1}$.

Observation 2: There a kernel $\psi \in L^\infty(K \times K)$ such that

$$A^{n-1}_w f^{(n)}(x) = \int_K \psi(x,y) \Delta^k \big(A^{n-1}_w f(y) \big) d\mu(y),$$

for any $f \in H^{2k}(K)$ and $n \geq 1$.

Proof of Observation 2. We only need to choose

$$\psi(x,y)=(-1)^k\int_{K^{k-1}}G^{(1)}_{y_1}(x)G(y_1,y_2)\cdots G(y_{k-1},y)d\mu(y_1)\cdots d\mu(y_{k-1}),$$

where $G_y(x)=G(x,y)$ is the Green's function. By Step 2, we immediately have $\psi\in L^\infty(K\times K)$. Since $h^{(1)}=0,\forall h\in\mathcal{H}_{k-1}$, we can easily see that $f^{(1)}(x)=\int_K\psi(x,y)\Delta^k f(y)d\mu(y)$. For $n\geq 2$, we use Observation 1.

Now, by using Observation 2 and Lemma 3.8, we can see that $\{f^{(n)}\}_{n\geq 0}\in l^2(L^\infty(K);r_w^k\mu_w^{k-1/2})$ for any $f\in H^{2k}(K)$. Then, a same proof as in Proposition 3.16 shows that $\sum_{n=0}^\infty f^{(n)}$ converges in $H^{2k}(K)$ with $\|\sum_{n=0}^\infty f^{(n)}\|_{H^{2k}(K)}\lesssim\|f\|_{H^{2k}(K)}$. On the other hand, we have $P_{S_{k,w}}f=\sum_{n=0}^\infty f^{(n)},\forall f\in H^{2k}(K)$ as desired. In fact, this is true if $A_w^l f\in\mathcal{H}_{k-1}$ for some $l\geq 0$, and this kind of functions are dense in $H^{2k}(K)$. □

6.3. **Proof of Theorem 6.2.** We return to prove Theorem 6.2. As shown in Lemma 6.4, the key is to construct the map "L". Theorem 6.7 will play a crucial role.

Lemma 6.8. *For each $w\in W_*$ and $l\geq 0$, there is a polynomial p_w such that $p_w(A_w)+id=0$ and $p_w(\mu_w^{-1}A_w^{-1})-id=0$ on $\tilde{E}_{l,w}$.*

Proof. Since $\tilde{E}_{l,w}$ is of finite dimension, there are polynomials p_1 and p_2 such that $p_1(A_w)=0$ and $p_2(\mu_w^{-1}A_w^{-1})=0$ on $\tilde{E}_{l,w}$. The zeros of p_1 and p_2 can be disjoint, since they can be just the eigenvalues of A_w and $\mu_w^{-1}A_w^{-1}$ respectively. Then p_1 and p_2 are coprime polynomials, and thus there exist polynomials r_1 and r_2 such that $r_1p_1-r_2p_2=1$. Then the polynomial $p_w=r_1p_1+r_2p_2$ will satisfy the requirement of the lemma. □

Definition 6.9. *(a). Let $k\geq 1$, $\omega=\tau\dot{w}\in\mathcal{P}$ and $l=l_{\omega(2k)}$. Take p_w as in Lemma 6.8. Define $L^{(k)}_\omega=\mu_\tau A_\tau^*P_{S_{k,w}}p_w(A_w)A_\tau$, where A_τ^* is the adjoint operator of A_τ.*

(b). Define $L^{(k)}=\sum_{\omega\in\mathcal{P}}L^{(k)}_\omega$.

We have the following Proposition.

Proposition 6.10. *$L^{(k)}+id\in\mathcal{L}\big(H^0(K),H^0(K)\big)\cap\mathcal{L}\big(H^{2k}(K),H^{2k}_0(K)\big)$ and $(L^{(k)})^*-id\in\mathcal{L}\big(H^0(K),H^0(K)\big)\cap\mathcal{L}\big(H^{2k}(K),H^{2k}_0(K)\big)$, where id is the identity map.*

Proof. Let $\omega=\tau\dot{w}\in\mathcal{P}$. For any $0\leq s<\infty$ and $f\in H^{2k}(K)$, by using Theorem 6.7, we always have $\mu_\tau A_\tau^*P_{S_{k,w}}A_w^sA_\tau f\in H^{2k}(K)$, has 0 tangent on $\mathcal{P}\setminus\{\pi(\omega)\}$, and

$$T^{(2k)}_\omega(\mu_\tau A_\tau^*P_{S_{k,w}}A_w^sA_\tau f)=T^{(2k)}_\omega(\mu_\tau A_\tau^*A_w^sA_\tau f)=A_w^s\big(T^{(2k)}_\omega(f)\big).$$

As a consequence, we see that $L^{(k)}_\omega f$ has 0 tangent at $\mathcal{P}\setminus\{\pi(\omega)\}$, and $T^{(2k)}_\omega(L^{(k)}_\omega f)=p_w(A_w)T^{(2k)}_\omega(f)$. By Definition 6.9 and Lemma 6.8, we conclude that $(L^{(k)}+id)f\in H^{2k}_0(K)$ by using the characterization of $H^{2k}_0(K)$ in Theorem 4.2.

To show the other half, we need some observations.

1) $\big(L^{(k)}_\omega\big)^*=\mu_\tau A_\tau^*p_w(A_w^*)P_{S_{k,w}}A_\tau$, and $\big(L^{(k)}\big)^*=\sum_{\omega\in\mathcal{P}}\big(L^{(k)}_\omega\big)^*$.

2) For $f\in H^{2k}(K)$, $T^{(2k)}_\omega(A_w^*f)=\mu_w^{-1}A_w^{-1}\big(T^{(2k)}_\omega(f)\big)$, noticing that $A_w^*(f)=\mu_w^{-1}f\circ F_w^{-1}$.
The rest of the proof is similar to that of the first part. □

Proof of Theorem 6.2. By using Lemma 6.4 and Proposition 6.10, we see that

$$[H^{-2k}(K), H^{2k}(K)]_{1/2} = H^0(K).$$

Thus for $-\infty < \sigma_1 < 0 < \sigma_2 < +\infty$ and $\theta = \frac{\sigma_1}{\sigma_1 - \sigma_2}$, by using Theorem 5.4 and 5.7, we get

$$\big[\big(H_{00}^{-\sigma_1}(K)\big)', H^{\sigma_2}(K)\big]_\theta = H^0(K).$$

As a consequence, we then have

$$[H^{\sigma_1}(K), H^{\sigma_2}(K)]_\theta = H^0(K),$$

because

$$H^0(K) = [H^{\sigma_1}(K), H_0^{\sigma_2}(K)]_\theta \subset [H^{\sigma_1}(K), H^{\sigma_2}(K)]_\theta \subset \big[\big(H_{00}^{-\sigma_1}(K)\big)', H^{\sigma_2}(K)\big]_\theta = H^0(K).$$

Then the theorem follows immediately. □

7. Examples

In this section, we present some concrete examples.

7.1. **D3-symmetric fractals.** Tangents on D3-symmetric p.c.f. self-similar sets have been studied in detail in [37] on the domain of Δ^k. Also see some related studies in [10, 9, 36].

More precisely, let's look at a p.c.f. self-similar set K with exactly three boundary points $V_0 = \{p_1, p_2, p_3\}$ such that $\pi^{-1}(p_i) = \dot{i}$. Assume that there exists a group $\mathcal{G}$ of homeomorphisms of K isomorphic to the D3-symmetric group that acts as permutations on V_0, and $\mathcal{G}$ preserves the harmonic structure and the self-similar measure of K. See Figure 1 for examples.

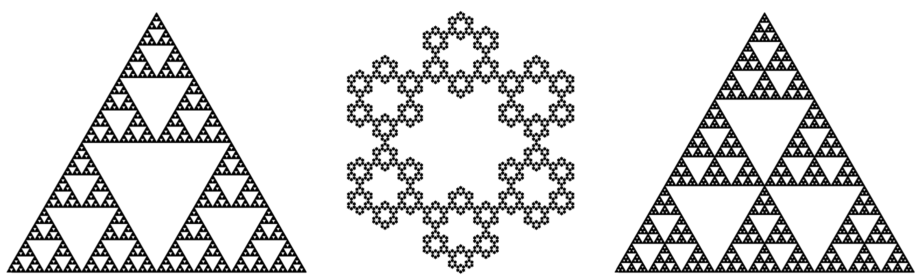

Figure 1. Examples of D3-symmetric p.c.f. self-similar sets: the Sierpinski gasket, the Hexagasket and the level-3 Sierpinski gasket.

For fixed $i \in \{1, 2, 3\}$, let h_T be the antisymmetric harmonic function with the boundary values $h_T(p_i) = 0, h_T(p_{i+1}) = 1, h_T(p_{i+2}) = -1$, where we use the cyclic notation $p_4 = p_1$. Then it is easy to see that,

$$\{\lambda_{l,i}\}_{l\geq 0} = \{r_i^n\mu_i^n, r_i^{n+1}\mu_i^n, \iota_i r_i^n\mu_i^n\}_{n\geq 0},$$

where ι_i is defined by the identity $A_i h_T = \iota_i h_T$. This means that $\sigma(A_i, \mathcal{H}_0) = \{1, r_i, \iota_i\}$.

In convention, we define *normal derivatives* and *tangential derivatives* of functions at p_i by the following pointwise formulas, if the limits exist,

$$\begin{cases} \partial_n f(p_i) = \lim\limits_{n\to\infty} r_i^{-n}\big(2f(p_i) - f(F_i^n p_{i+1}) - f(F_i^n p_{i+2})\big), \\ \partial_T f(p_i) = \lim\limits_{n\to\infty} \iota_i^{-n}\big(f(F_i^n p_{i+1}) - f(F_i^n p_{i+2})\big). \end{cases}$$

Assuming (**C2**), by using Theorem 3.14, we can easily see the following result.

Theorem 7.1. *(a). For* $\sigma > 2n + 2 - \frac{d_S}{2}$*,* $n \in \mathbb{Z}^+$*,* $\partial_n \Delta^n f(p_i)$ *is well defined,* $\forall f \in H^\sigma(K)$*.*
(b). For $\sigma > 2n + \frac{2\log \iota_i}{(1+d_H)\log r_i} + \frac{d_S}{2}$*,* $n \in \mathbb{Z}^+$*,* $\partial_T \Delta^n f(p_i)$ *is well defined,* $\forall f \in H^\sigma(K)$*.*

The following is an equivalent narration of Theorem 4.2.

Theorem 7.2. *For* $\sigma \geq 0$ *and* $f \in H^\sigma(K)$*, we have* $f \in H_0^\sigma(K)$ *if and only if*

$$\begin{cases} \Delta^n f(p_i) = 0, & \forall 0 \leq n < \frac{\sigma}{2} - \frac{d_S}{4} \text{ and } i = 1,2,3, \\ \partial_n \Delta^n f(p_i) = 0, & \forall 0 \leq n < \frac{\sigma}{2} + \frac{d_S}{4} - 1 \text{ and } i = 1,2,3, \\ \partial_T \Delta^n f(p_i) = 0, & \forall 0 \leq n < \frac{\sigma}{2} - \frac{d_S}{4} - \frac{\log \iota_i}{(1+d_H)\log r_i} \text{ and } i = 1,2,3. \end{cases}$$

7.2. **The Vicsek set.** Let $\{p_i\}_{i=1}^4$ be the four vertices of a unit square, and p_5 be the center of the square. The Vicsek set $\mathcal{V}$ (see Figure 2) is the attractor of the i.f.s. $\{F_i\}_{i=1}^5$, where

$$F_i x = \frac{1}{3}x + \frac{2}{3}p_i, \text{ for } i = 1,2,3,4,5.$$

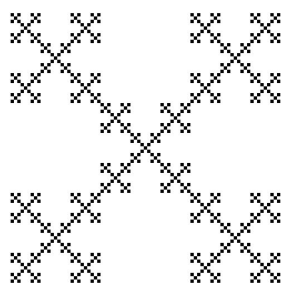

FIGURE 2. The Vicsek set $\mathcal{V}$.

The boundary set of $\mathcal{V}$ is $V_0 = \{p_i\}_{i=1}^4$ with $\pi^{-1}(p_i) = \dot{i}$. There is a unique S4-symmetric harmonic structure on $\mathcal{V}$, with

$$r_i = \frac{1}{3}, i = 1,2,3,4,5,$$

and

$$\mathcal{E}_0(f,g) = \sum_{i \neq j} \big(f(p_i) - f(p_j)\big)\big(g(p_i) - g(p_j)\big), \quad \forall f,g \in l(V_0).$$

In addition, we take μ to be the canonical normalized Hausdorff measure on $\mathcal{V}$.

The Vicsek set $\mathcal{V}$ is an interesting example in that $\{\lambda_{l,i}\}_{l\geq 0} = \{15^{-n}, 3^{-1}\cdot 15^{-n}\}_{n\geq 0}$, with each $\lambda_{l,i}$ has a one dimensional generalized eigenspace. We have the following narration of Theorem 4.2.

Theorem 7.3. *For* $\sigma \geq 0$ *and* $f \in H^\sigma(\mathcal{V})$*, we have* $f \in H_0^\sigma(\mathcal{V})$ *if and only if*

$$\begin{cases} \Delta^n f(p_i) = 0, & \forall 0 \leq n < \frac{\sigma}{2} - \frac{d_S}{4} \text{ and } i = 1,2,3,4, \\ \partial_n \Delta^n f(p_i) = 0, & \forall 0 \leq n < \frac{\sigma}{2} + \frac{d_S}{4} - 1 \text{ and } i = 1,2,3,4. \end{cases}$$

Furthermore, for $\sigma \geq 0$, write $H_D^\sigma(\mathcal{V}) = (id - \Delta_D)^{-\sigma/2}L^2(\mathcal{V})$ and $H_N^\sigma(\mathcal{V}) = (id - \Delta_N)^{-\sigma/2}L^2(\mathcal{V})$, where Δ_D and Δ_N are the Dirichlet and Neumann Laplacians. See [11, 38] for a detailed discussion on these spaces. Then we have

Theorem 7.4. *For $\sigma \geq 0$, $H^\sigma_{00}(\mathcal{V}) = H^\sigma_D(\mathcal{V}) \cap H^\sigma_N(\mathcal{V})$ with $\|f\|_{H^\sigma_{00}(\mathcal{V})} \asymp \|f\|_{H^\sigma_D(\mathcal{V})} + \|f\|_{H^\sigma_N(\mathcal{V})}$.*

Proof. Fix $k \geq \lceil \sigma/2 \rceil$, we break the spaces $\mathcal{H}_{k-1}$ into two parts, $\mathcal{H}_{k-1} = X^{(i)} \oplus Y^{(i)}$ such that $\sigma(A_i, X^{(i)}) = \{1, 15^{-1}, \cdots, 15^{-k+1}\}$ and $\sigma(A_i, Y^{(i)}) = \{3^{-1}, 45^{-1}, \cdots, 3^{-1} \cdot 15^{-k+1}\}$.

Using the notations in Definition 3.4 and 4.3, one can check that

$$\begin{cases} H^\sigma_D(\mathcal{V}) = \ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}} \oplus \Big(\oplus_{i=1}^4 \mathcal{R}_{\mathcal{H}_{k-1},i}\big(l^2(X^{(i)}; r_i^{\sigma/2}\mu_i^{(\sigma-1)/2})\big)\Big) \\ \qquad\qquad\qquad\qquad\quad \oplus \Big(\oplus_{i=1}^4 \mathcal{R}_{\mathcal{H}_{k-1},i}\big(l^2(Y^{(i)}, A_i; r_i^{\sigma/2}\mu_i^{(\sigma-1)/2})\big)\Big), \\ H^\sigma_N(\mathcal{V}) = \ker_\sigma \mathcal{T}_{\mathcal{H}_{k-1}} \oplus \Big(\oplus_{i=1}^4 \mathcal{R}_{\mathcal{H}_{k-1},i}\big(l^2(X^{(i)}, A_i; r_i^{\sigma/2}\mu_i^{(\sigma-1)/2})\big)\Big) \\ \qquad\qquad\qquad\qquad\quad \oplus \Big(\oplus_{i=1}^4 \mathcal{R}_{\mathcal{H}_{k-1},i}\big(l^2(Y^{(i)}; r_i^{\sigma/2}\mu_i^{(\sigma-1)/2})\big)\Big). \end{cases}$$

The theorem follows immediately. □

Appendix A. The proof of Lemma 3.6

In this appendix, we prove Lemma 3.6. Recall that X is a Banach space with norm $\|\cdot\|_X$, $A : X \to X$ is a compact operator, and $\sigma(A, X)$ is the spectrum of $A : X \to X$. We let $\{\lambda_l\}_{l\geq 0}$ be the nonzero eigenvalues of A, which is ordered in decreasing order of absolute values. Let E_l be the corresponding generalized eigenspaces, and denote $\tilde{E}_l = \oplus_{i=0}^{l''} E_i$ and $\hat{E}_l = \oplus_{i=l'}^{l''} E_i$, where $l' = \min\{i \geq 0 : |\lambda_i| = |\lambda_l|\}, l'' = \max\{i \geq 0 : |\lambda_i| = |\lambda_l|\}$.

Proof of Lemma 3.6. Let $\boldsymbol{s} \in l^2(X, A; \alpha)$, and denote $t_0 = s_0$ and $t_n = s_n - A s_{n-1}$ for $n \geq 1$. Clearly, $\boldsymbol{t} := \{t_n\}_{n\geq 0} \in l^2(X; \alpha)$ with $\|\boldsymbol{t}\|_{l^2(X;\alpha)} \asymp \|\boldsymbol{s}\|_{l^2(X,A;\alpha)}$, and also $s_n = \sum_{m=0}^n A^{n-m} t_m$.

We consider three cases separately.

Case 1: $\alpha > |\lambda_0|$ or $\sigma(A, X) = \{0\}$.

Using the Minkowski inequality, noticing that α is larger than the spectral radius of $A : X \to X$, we get

$$\begin{aligned} \|\boldsymbol{s}\|_{l^2(X;\alpha)} &= \Big\| \sum_{m=0}^n A^{n-m} t_m \Big\|_{l^2(X;\alpha)} \leq \Big\| \sum_{m=0}^n \alpha^{-n} \|A^m t_{n-m}\|_X \Big\|_{l^2} \\ &\leq \Big\| \sum_{m=0}^\infty 1_{n\geq m} \alpha^{m-n} \|\alpha^{-m} A^m t_{n-m}\|_X \Big\|_{l^2} \leq \sum_{m=0}^\infty \|\alpha^{-m} A^m\|_{X\to X} \cdot \|\boldsymbol{t}\|_{l^2(X;\alpha)} \\ &\lesssim \|\boldsymbol{s}\|_{l^2(X,A;\alpha)}. \end{aligned}$$

The other direction $\|\boldsymbol{s}\|_{l^2(X,A;\alpha)} \lesssim \|\boldsymbol{s}\|_{l^2(X;\alpha)}$ is obvious. To conclude, in this case, we have $l^2(X, A; \alpha) = l^2(X; \alpha) = \overline{l^2(X; \alpha)}$.

Case 2: $|\lambda_{l+1}| < \alpha < |\lambda_l|$ or $\alpha < |\lambda_l| = \min\{|\lambda_k| : \lambda_k \in \sigma(A, X)\}$.

First, we assume A is of finite rank and $X = \oplus_{i=0}^l E_i$, so that A^{-1} is well defined. Clearly, the following limit exists in X,

$$s_{\lim} = \lim_{n\to\infty} A^{-n} s_n = \sum_{m=0}^\infty A^{-m} t_m,$$

since α^{-1} is larger than the spectral radius of A^{-1}. Thus

$$s_n - A^n s_{\lim} = -\sum_{m=n+1}^{\infty} A^{n-m} t_m.$$

Now we define $\boldsymbol{s}' = \{s_n - A^n s_{\lim}\}_{n\geq 0}$. Using Minkowski inequality, we get

$$\begin{aligned}\|\boldsymbol{s}'\|_{l^2(X;\alpha)} &= \big\|\alpha^{-n}\| \sum_{m=n+1}^{\infty} A^{n-m} t_m\|_X\big\|_{l^2}\\ &\leq \big\|\sum_{m=1}^{\infty} \alpha^{-n-m}\|\alpha^m A^{-m} t_{n+m}\|_X\big\|_{l^2} \leq \sum_{m=1}^{\infty}\|\alpha^m A^{-m}\|_{X\to X}\cdot\|\boldsymbol{t}\|_{l^2(X;\alpha)}\\ &\lesssim \|\boldsymbol{s}\|_{l^2(X,A;\alpha)}.\end{aligned}$$

This shows that $\boldsymbol{s}' \in l^2(X;\alpha)$ with the estimate of the norm. So we have $\boldsymbol{s} = \mathcal{S}_A(s_{\lim}) + \boldsymbol{s}'$, with $\boldsymbol{s}' \in l^2(X;\alpha)$. Clearly, both $\mathcal{S}_A(X)$ and $l^2(X;\alpha)$ are closed subspace of X, and $\mathcal{S}_A(X)\cap l^2(X;\alpha) = \{0\}$. So we have proved the decomposition

$$l^2(X, A;\alpha) = \mathcal{S}_A(\tilde{E}_l) \oplus l^2(X;\alpha).$$

For general case, since A admits a discrete spectrum, we can find a closed subspace $\tilde{E}_l^\circ$ such that $\sigma(A, \tilde{E}_l^\circ) = \sigma(A, X)\setminus\{\lambda_i\}_{i=0}^l$, and $X = \tilde{E}_l \oplus \tilde{E}_l^\circ$. Then we see that

$$\begin{aligned}l^2(X, A;\alpha) &= l^2(\tilde{E}_l, A;\alpha) \oplus l^2(\tilde{E}_l^\circ, A;\alpha)\\ &= \mathcal{S}_A(\tilde{E}_l) \oplus l^2(\tilde{E}_l;\alpha) \oplus l^2(\tilde{E}_l^\circ;\alpha) = \mathcal{S}_A(\tilde{E}_l)\oplus l^2(X;\alpha),\end{aligned}$$

where we use (a) and the finite rank case that we have proved. Obvious, this implies that $\overline{l^2(X;\alpha)} = l^2(X;\alpha)$ in this case, and we have the desired decomposition in part (b) of the lemma.

Case 3: $\alpha = |\lambda_l|$ for some $l \geq 0$. (assume $|\lambda_l| > |\lambda_{l-1}|$ if $l \geq 1$)

We first show that $\mathcal{S}_A(\hat{E}_l) \subset \overline{l^2(X;\alpha)}$. Let $\boldsymbol{s} = \mathcal{S}_A(s)$ for some $s \in E_l$. There is a $d \geq 0$ such that $(A-\lambda_l)^d s \neq 0$ and $(A-\lambda_l)^{d+1}s = 0$. Write $s^{(k)} = (A-\lambda_l)^k s, 0 \leq k \leq d$. Fix $m_0, m_1, \cdots, m_d \in \mathbb{N}$ and take $M_k = \sum_{i=0}^k m_i$ (set $M_{-1} = 0$). Then we can design a sequence $\boldsymbol{s}^{(m_0,m_1,\cdots,m_d)} = \{s_n^{(m_0,m_1,\cdots,m_d)}\}_{n\geq 0}$ in $l^2(X;\alpha)$ as follows.

$$\lambda_l^{-n} s_n^{(m_0,m_1,\cdots,m_d)} = \begin{cases} s, & \text{if } n = 0,\\ \lambda_l^{-n} A s_{n-1}^{(m_0,m_1,\cdots,m_d)} - m_k^{-1} a_k s^{(k)}, & \text{if } M_{k-1} < n \leq M_k,\\ \ \text{where } \lambda_l^{-M_{k-1}} s_{M_{k-1}}^{(m_0,m_1,\cdots,m_d)} = a_k s^{(k)} + b_{k+1}s^{(k+1)} + \cdots + b_d s^{(d)}, & \\ 0, & \text{if } n > M_d.\end{cases}$$

We can easily check that

$$\lim_{m_0\to\infty}\lim_{m_1\to\infty}\cdots\lim_{m_d\to\infty}\|\mathcal{S}_A(s) - \boldsymbol{s}^{(m_0,m_1,\cdots,m_d)}\|_{l^2(X,A;\alpha)} = 0,$$

which gives that $\mathcal{S}_A(E_l) \subset \overline{l^2(X;\alpha)}$. Similarly, we have $\mathcal{S}_A(\hat{E}_l) \subset \overline{l^2(X;\alpha)}$.

As a consequence, we have $\overline{l^2(X;\alpha)} \neq l^2(X;\alpha)$, since $S_A(\hat{E}_l) \subset \overline{l^2(X;\alpha)} \setminus l^2(X;\alpha)$. It remains to prove the decomposition $l^2(X,A;\alpha) = \mathcal{S}_A(\tilde{E}_{l-1}) \oplus \overline{l^2(X;\alpha)}$. Using the decomposition we already proved in Case 1 and 2, we can see

$$\mathcal{S}_A(\tilde{E}_{l-1}) + \overline{l^2(X;\alpha)} \supset \mathcal{S}_A(\tilde{E}_l) + l^2(X;\alpha) \supset l^2(X,A;\alpha-\epsilon),$$

for some small $\epsilon > 0$, where we set $\tilde{E}_{-1} = \{0\}$ for convenience. Since $l^2(X,A;\alpha-\epsilon)$ is a dense subspace of $l^2(X,A;\alpha)$, we have

$$\mathcal{S}_A(\tilde{E}_{l-1}) + \overline{l^2(X;\alpha)} = \overline{\mathcal{S}_A(\tilde{E}_{l-1}) + \overline{l^2(X;\alpha)}} \supset l^2(X,A;\alpha). \tag{A.1}$$

Lastly, we have

$$\mathcal{S}_A(\tilde{E}_{l-1}) \cap \overline{l^2(X;\alpha)} \subset \mathcal{S}_A(\tilde{E}_{l-1}) \cap \overline{l^2(X;\alpha+\varepsilon)} = \mathcal{S}_A(\tilde{E}_{l-1}) \cap l^2(X;\alpha+\varepsilon) = \{0\}, \tag{A.2}$$

for some small $\varepsilon > 0$. The decomposition $l^2(X,A;\alpha) = \mathcal{S}_A(\tilde{E}_{l-1}) \oplus \overline{l^2(X;\alpha)}$ follows immediately from (A.1) and (A.2). □

Appendix B. On the condition (C1)

Although a large amount of p.c.f. fractals, including all the examples in Section 7, satisfy (**C1**), there do exist counter examples.

Example. Let $\{p_1,p_2,p_3\}$ be the three vertices of a triangle, and $p_4 = \frac{1}{3}\sum_{i=1}^3 p_i$ be the center. We define an i.f.s. $\{F_i\}_{i=1}^4$ by

$$F_i(x) = \frac{1}{2}x + \frac{1}{2}p_i, i = 1,2,3,$$
$$F_4(x) = \frac{1}{4}x + \frac{3}{4}p_4.$$

Call the unique compact set, denoted by $\mathcal{SG}_f$, satisfying $\mathcal{SG}_f = \bigcup_{i=1}^4 F_i(\mathcal{SG}_f)$, the *filled Sierpinski gasket* [2]. See Figure 3.

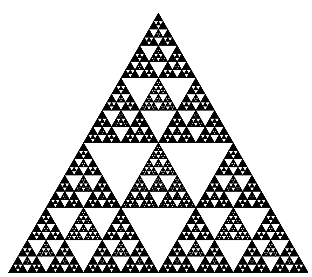

Figure 3. The filled Sierpinski gasket $\mathcal{SG}_f$.

One can check that $\mathcal{C} = \{1\dot{2}, 1\dot{3}, 2\dot{1}, 2\dot{3}, 3\dot{1}, 3\dot{2}, 4\dot{1}, 4\dot{2}, 4\dot{3}, 12\dot{3}, 13\dot{2}, 21\dot{3}, 23\dot{1}, 31\dot{2}, 32\dot{1}\}$, $\mathcal{P} = \{\dot{1}, \dot{2}, \dot{3}, 1\dot{2}, 1\dot{3}, 2\dot{1}, 2\dot{3}, 3\dot{1}, 3\dot{2}\}$ and $V_0 = \{p_1,p_2,p_3,F_1p_2,F_2p_3,F_3p_1\}$. One can see that

$$\pi^{-1}(F_1p_2) = \{1\dot{2}, 2\dot{1}\}.$$

As a consequence, (**C1**) fails for $\mathcal{SG}_f$. □

Fortunately, **all the main theorems** in this paper, including Theorem 3.14, 3.17, 4.2, 5.4, 5.6, 5.7 and 6.2, are valid even if (**C1**) is not satisfied, as long as we assume (**C2**). Clearly, we did not use (**C1**) in Section 3, but Section 4 and Section 5 are somewhat delicate, where we need a precise description of the pre-tangents at the boundary. Below we briefly show the necessary materials in proving Theorem 4.2, 5.4, 5.6, 5.7 and 6.2 without using (**C1**).

We need some new notations. Let $\omega = \dot{w} \in \mathcal{P}$ and $k \in \mathbb{N}$.

Notation 1: *Denote* $\tilde{A}_w : \mathbb{C}^k \to \mathbb{C}^k$ *by* $(x_0, x_1, \cdots, x_{k-1}) \to (x_0, r_w\mu_w x_1, \cdots, (r_w\mu_w)^{k-1}x_{k-1})$.

Notation 2: *Denote* $\mathcal{H}_{k-1}^{(w)} = \{h \in \mathcal{H}_{k-1} : \Delta^l h(\pi(\dot{w})) = 0, \forall 0 \le l \le k-1\}$.

With some effort, one can check the following claims.

Claim 1: *There is a natural isomorphism* $\mathbb{C}^k \times \mathcal{H}_{k-1}^{(w)} \to \mathcal{H}_{k-1}$*, which gives us natural isomorphisms* $I_w : l^2(\mathbb{C}^k, \tilde{A}_w; \alpha) \times l^2(\mathcal{H}_{k-1}^{(w)}, A_w; \alpha) \to l^2(\mathcal{H}_{k-1}, A_w; \alpha)$.

Claim 2: *Let* $\sigma \ge 0$ *and* $k \ge \lceil \sigma/2 \rceil$, $p \in V_0$, $\pi^{-1}(p) = \{\omega_1, \cdots, \omega_m\} = \{\tau_1\dot{w}_1, \cdots, \tau_m\dot{w}_m\}$ *and*

$$X_{p,\sigma} = \big\{\big(\{\boldsymbol{x}_n^{(1)}\}_{n\ge0}, \{\boldsymbol{x}_n^{(2)}\}_{n\ge0}, \cdots, \{\boldsymbol{x}_n^{(m)}\}_{n\ge0}\big) \in \prod_{i=1}^m l^2(\mathbb{C}^k, \tilde{A}_{w_i}; r_{w_i}^{\sigma/2}\mu_{w_i}^{(\sigma-1)/2}) :$$

$$\lim_{n\to\infty}(r_{\tau_i}\mu_{\tau_i})^{-l}(r_{w_i}\mu_{w_i})^{-nl}(\boldsymbol{x}_n^{(i)})_l = \lim_{n\to\infty}(r_{\tau_j}\mu_{\tau_j})^{-l}(r_{w_j}\mu_{w_j})^{-nl}(\boldsymbol{x}_n^{(j)})_l, \forall i \ne j, 0 \le l \le k-1\big\}.$$

There is an isomorphism

$$I_p : X_{p,\sigma} \to l^2(\mathbb{C}^k, \tilde{A}_{w_1}; r_{w_1}^{\sigma/2}\mu_{w_1}^{(\sigma-1)/2}) \times \big(\prod_{i=2}^m l^2(\mathbb{C}^k; r_{w_i}^{\sigma/2}\mu_{w_i}^{(\sigma-1)/2})\big),$$

defined consistently for all $\sigma \ne 2l + \frac{d_S}{2}, 0 \le l \le k-1$.

The key steps of constructing I_p is: first we pick a nondecreasing sequence $\{\ell_n\}_{n\ge0}$ such that $r_{w_1}^{\ell_n} \asymp r_{w_j}^n$, and hence by (**C2**) $\mu_{w_1}^{\ell_n} \asymp \mu_{w_j}^n$; next we define a sequence $\{\hat{\boldsymbol{x}}_n^{(j)}\}_{n\ge0}$ by

$$A_{\tau_j}^{-1}\big(\tilde{A}_{w_j}^{-n-1}\hat{\boldsymbol{x}}_{n+1}^{(j)} - \tilde{A}_{w_j}^{-n}\hat{\boldsymbol{x}}_n^{(j)}\big) = A_{\tau_1}^{-1}\big(\tilde{A}_{w_1}^{-\ell_{n+1}}\boldsymbol{x}_{\ell_{n+1}}^{(1)} - \tilde{A}_{w_1}^{-\ell_n}\boldsymbol{x}_{\ell_n}^{(1)}\big).$$

It is easy to check that $\{\hat{\boldsymbol{x}}_n^{(j)}\}_{n\ge0} \in l^2(\mathbb{C}^k, \tilde{A}_{w_j}; r_{w_j}^{\sigma/2}\mu_{w_j}^{(\sigma-1)/2})$ and $\{\boldsymbol{x}_n^{(j)} - \hat{\boldsymbol{x}}_n^{(j)}\}_{n\ge0} \in l^2(\mathbb{C}^k; r_{w_j}^{\sigma/2}\mu_{w_j}^{(\sigma-1)/2})$ using Lemma 3.6 if $\big(\{\boldsymbol{x}_n^{(1)}\}_{n\ge0}, \{\boldsymbol{x}_n^{(2)}\}_{n\ge0}, \cdots, \{\boldsymbol{x}_n^{(m)}\}_{n\ge0}\big) \in X_{p,\sigma}$ and $\sigma \ne 2l + \frac{d_S}{2}, \forall l \ge 0$. The rest of the construction is easy and left to the reader.

By applying the isomorphisms in Claim 1 and Claim 2, we finally are able to give a neat description of pre-tangents for $H^{2l}(K), 0 \le l \le k$. Theorem 5.4 is true since we can still show that the space of pre-tangents is stable under complex interpolation. To show Theorem 4.2, a similar argument as Theorem 4.7 is enough, noticing that we did not use (**C1**) in the proof of Lemma 4.6. The definition of $H_{00}^\sigma(K)$ remains the same even if (**C1**) is not satisfied, so Theorem 5.6 remains the same. We may take $\tilde{H}_0^\sigma(K)$ as the right side of (4.1), then we can see that $H_{00}^\sigma(K) \subset H_0^\sigma(K) \subset \tilde{H}_0^\sigma(K)$, and $[\tilde{H}_{\sigma_1}^0(K), \tilde{H}_{\sigma_2}^0(K)]_\theta = H_{00}^\sigma(K)$ with $\sigma = (1-\theta)\sigma_1 + \theta\sigma_2$. Theorem 5.7 then follows as well using Theorem 5.6. Lastly, since Theorem 6.2 is a consequence of the above theorems, it remains valid.

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Department of Mathematics, Cornell University, Ithaca 14853, USA
Email address: sc2873@cornell.edu

Department of Mathematics, Nanjing University, Nanjing 210093, China
Email address: huaqiu@nju.edu.cn