HOMOGENEOUS DIRICHLET FORMS ON P.C.F. FRACTALS AND THEIR SPECTRAL ASYMPTOTICS

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Abstract. We formulate a class of "homogeneous" Dirichlet forms (DF) that aims to explore those forms that do not satisfy the conventional energy self-similar identity (degenerated DFs). This class of DFs has been studied in [12, 13, 16, 17] in connection with the asymptotically one-dimensional diffusions on the Sierpinski gaskets (SG) and their generalizations. In this paper, we give a systematic study of such DFs and their spectral property. We also emphasis the construction of some new homogeneous DFs. A basic assumption on the resistance growth in [13] to investigate the heat kernel and the existence of the "non-fixed point" limiting diffusion is answered analytically.

Contents

1. Introduction

Dirichlet forms play a central role in the analysis on fractals. There is a large literature on the topic based on Kigami's analytic approach on *post-critically finite (p.c.f.) self-similar sets*, and the probabilistic approach of Lindstrøm on nested fractals as well as Barlow and Bass on the Sierpinski carpet (see [1, 2, 9, 14, 15, 18, 20, 21, 23, 24, 25] and the references therein).

On a symmetric p.c.f. self-similar set K (nested fractal), the most celebrated and well-studied structure is the *energy self-similar identity*, which is a consequence of a regular harmonic structure that can be obtained by a (non-constructive) fixed point of certain nonlinear trace map of the resistances from V_1 to V_0 (V_0 is the boundary of the p.c.f. set, V_1 is its first iteration by the IFS) [21]. By equipping with a self-similar measure μ, the energy

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form defines a local regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$, which yields a Laplacian. One of the important questions on $(\mathcal{E},\mathcal{F})$ concerns the distribution of the eigenvalues. In [22], Kigami and Lapidus proved that on the p.c.f. self-similar sets that admit an energy self-similar identity, the eigenvalue counting function $\rho(x)$ of the associated Laplacian satisfies the estimate

$$\rho(x) \asymp x^{d_S/2}, \quad x \to \infty$$

where d_S is the spectral dimension as the unique solution of $\sum_{i=1}^{N}(r_i\mu_i)^{d_S/2} = 1$, with r_i being the resistance factor in the energy self-similar identity, and μ_i being the weight of the self-similar measure. (We use $f \asymp g$ to mean $C^{-1}f \leq g \leq Cf$ for some $C > 0$, on the variables of f and g.)

On the other hand, much less is known for Dirichlet forms that do not satisfy the energy self-similar identity, the DFs that do not come from the fixed point approach of Kigami [21] (degenerated DFs). To the authors' knowledge, the earliest work on such forms can be traced back to Hattori, Hattori and Watanabe [17]. They constructed a sequence of compatible networks of "one-parameter family" of conductances on the approximating graphs of the Sierpinski gasket (SG), which converges asymptotically to a Brownian motion (BM), locally confined to the direction of largest growth rate of conductances as the scale is reduced, and globally still behave like a BM. The heat kernel and the spectral distribution of the Laplacian were further investigated by Hambly and Jones in [12]. Furthermore Hambly and Kumagai [13] extended this consideration to some class of generalized SG. These degenerated diffusion can also be viewed as "non-fixed point" diffusions compared with the traditional fixed point constructions of Dirichlet forms on p.c.f. self-similar sets. Recently, by adopting the technique of one-parameter family of conductance, Hambly and Yang [16] gave a further investigation of this type of diffusions on more general fractal sets, and studied the "limiting process" on the "limiting fractal" (usually non-p.c.f.) in the sense of Gromov-Hausdorff under the resistance metrics.

It is clear that direct analytic approaches can also provide flexible ways to construct degenerated energy forms. Recently, the authors gave a characterization of all the compatible networks on the 2-dim SG [11], which strengthened the construction in [17], and improved the spectral estimate in [12]. Also in [10], some new *asymmetric* p.c.f. fractals were introduced, and the connection of compatible sequence of resistances and the existence of local regular Dirichlet forms were investigated. In particular it was given constructive algorithms to obtain energy self-similar identities as well as degenerated DFs.

The goal of this paper is two-fold. First, by summing up the studies mentioned above, we formulate two conditions on the compatible sequence of conductances on the $V_n, n \geq 0$ (see Section 3): Condition (A1) is to assume that all the cells of the same level have the same conductance; it gives a *homogeneity* (or translation invariant) of the conductance in each level. Condition (A2) is such that for $p, q \in V_0, p \neq q$, the conductance $c_n(p,q)$ on the n-level has an exponential growth rate, i.e. $c_n(p,q) \asymp \kappa(p,q)^n$ for some constant $\kappa(p,q) > 1$, it allows anisotropic growth rate of the conductances.

In Section 3, we will give a detailed study of the compatible sequence of conductances with (A1) and (A2), and to obtain a resistance metric, which define a resistance form and a Dirichlet form (Corollary 3.3). We call this a homogeneous resistance/Dirichlet form. For the estimation of the eigenvalue counting function, we need an additional condition

(A3) (Section 4), and use a similar technique as in [22] to establish an estimation of the eigenvalue counting function (Theorem 4.6).

Our second goal is to investigate the constructive aspect of degenerated homogeneous resistance forms. We assume (A1) and look for a compatibility condition depending on the geometry of the underlying p.c.f. set. The basic approach (see [17]) is to determine the trace map Φ first. Then start on some initial conductance (or resistance) $\mathbf{x}_0$ on V_0, and use the inverse Φ^{-1} inductively to obtain a compatible sequence $\{\mathbf{x}_n\}_n$, and to verify it satisfies (A2) (call it inverse recursive construction). However there are technical difficulties arising in each step; there is no general criterion available yet, and is still a challenging question. We will illustrate this thought through the following examples.

The first degenerated DF is on an asymmetric "eyebolted" Vicsek cross (adding two eyebolts to destroy the symmetric) [10]. For this, we need to introduce a " ⊠-X transform" for network of four vertices to evaluate the resistances (analogous to the well-known Δ-Y transform for triangles) (Theorem 5.1). The second example is on the 3-dim SG (§5, Example 2). We consider certain initial conductances $\mathbf{x}_0$ on the resistance network of V_0 that give a "two-parameter family" of conductances, which has more complicated structure than the one-parameter family. Using a rather different proof, we are able to identify the region of $\mathbf{x}_0$ that (A2) is satisfied and give the explicit growth rates $\kappa(p,q)$. Eventually we show that there are only two types of homogeneous DFs, one is the standard DF, and the others are degenerated DFs resembles the type in [17]. (see Theorem 5.4(ii) and Remarks 2, 3, 4) and Section 7.

In [13], Hambly and Kumagai studied the d-dimensional ℓ-level SG K (also the Vicsek checkerboard), using the technique of the one-parameter family of conductance [17] to investigate the diffusion on K. They made an assumption on the growth rate of the conductances [13, p.373, Assumption $R_G\beta > 1$] to guarantee their models will have a local regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on K such that the domain $\mathcal{F}$ is contained in $C(K)$. They showed that in some cases the assumption holds, but it remains unclear whether it hold for all cases. In here, we use a shorting technique of resistance networks and proved that the assumption always hold in their generalized SG model (but fails in the Vicsek checkerboard case); moreover, we obtain the exact growth rates of the anisotropic conductances.

We organize the paper as follows. Section 2 on the resistance forms and the p.c.f. sets is just a summary of some facts in [21]. In Section 3, we study the compatible sequence of resistance forms with conditions (A1), (A2), and prove the estimation of the resistance metric. We then use this and an additional condition (A3) in Section 4 to conclude the spectral asymptotics of the homogeneous DFs. In Section 5, we examine in detail the construction of the homogeneous resistance forms on the 3-dim SG. Section 6 is an investigation on the basic assumption $R_G\beta > 1$ in [13]. Some remarks and open questions are also presented.

2. Preliminaries

This section summaries some facts from Kigami [21] (except Proposition 2.1), the knowledgeable reader can go directly to Section 3. Let X be a set, and $\ell(X)$ be the space of functions on X. A pair $(\mathcal{E},\mathcal{F})$ is called a *resistance form* on X if it satisfies the following conditions.

(RF1). $\mathcal{F}$ is a linear subspace of $\ell(X)$ containing constant functions, and $\mathcal{E}$ is a nonnegative symmetric bilinear form on $\mathcal{F}$. In addition, $\mathcal{E}(u) := \mathcal{E}(u,u) = 0$ if and only if u is constant.

(RF2). Let $\sim$ be an equivalence relation on $\mathcal{F}$ defined by $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.

(RF3). If $x \neq y$, then there is $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

(RF4). For any $p, q \in X$,

$$R(p,q) := \sup\left\{\frac{|u(p)-u(q)|^2}{\mathcal{E}(u)} \,:\, u \in \mathcal{F}, \mathcal{E}(u) > 0\right\} \tag{2.1}$$

is finite.

(RF5). For $u \in \mathcal{F}$, let $\bar{u} = (u \vee 0) \wedge 1$. Then $\bar{u} \in \mathcal{F}$, and $\mathcal{E}(\bar{u}) \leq \mathcal{E}(u)$.

The $R(p,q),\ p,q \in X$ in (2.1) actually defines a metric on X and is called an *effective resistance metric* on X. There is another useful equivalent expression of this metric:

$$R(p,q)^{-1} = \inf\{\mathcal{E}(u) : u \in \mathcal{F}, u(p) = 1, u(q) = 0\}. \tag{2.2}$$

Let $\{(X_n, E_n)\}_{n=0}^{\infty}$ be a sequence of resistance forms, where the X_n's are finite sets, $X_n \subset X_{n+1}$ for all $n \geq 0$ and the energy is given by

$$E_n(u) = \sum\nolimits_{p,q\in X_n} c_{pq}^{(n)}(u(p)-u(q))^2, \quad u \in \ell(X_n), \tag{2.3}$$

where $c_{pq}^{(n)} \geq 0$. We call these $c_{pq}^{(n)} > 0$ the *conductance* of p and q on (X_n, E_n). The sequence is called *compatible* if the restriction (or trace) of E_{n+1} on $\ell(X_n)$ equals to $(X_n, E_n),\ n \geq 0$. The pair (X_n, E_n) in (2.3) corresponds to a resistance network G_n with *resistance* $r_{pq}^{(n)} = (c_{pq}^{(n)})^{-1}$. An equivalent definition of compatibility on the sequence of networks $\{G_n\}_{n=0}^{\infty}$ is that the trace $P(G_n)$ on X_{n-1} equals $G_{n-1},\ n \geq 1$ [21].

Let $\{(X_n, E_n)\}_{n=0}^{\infty}$ be a compatible sequence. We have for any $n \geq 1$, $u \in \ell(X_{n-1})$,

$$\min\{E_n(v) : v \in \ell(X_n), v|_{X_{n-1}} = u\} = E_{n-1}(u).$$

We call the v that attains the minimum the harmonic extension of u. For $u_n \in \ell(X_n)$, by applying the harmonic extension inductively, we obtain a sequence $\{u_m\}_{m\geq n}$ of harmonic extensions on X_m with $E_m(u_m) = E_n(u_n)$.

For any function u on $X_* = \cup_n X_n$, we note that $\{E_n(u|_{X_n})\}_n$ is an increasing sequence, so that

$$\mathcal{E}(u) := \lim_{n\to\infty} E_n(u|_{X_n}) \tag{2.4}$$

exists (may equal $+\infty$). Define

$$\mathcal{F} = \{u \in \ell(X_*) :\ \mathcal{E}(u) < \infty\}.$$

Then $(\mathcal{E}, \mathcal{F})$ is a resistance form, which induces an effective resistance metric R on X_*. Denote by X the completion of X_* with respect to R. It follows that $(\mathcal{E}, \mathcal{F})$ extends to a resistance form on X [21, Theorem 2.3.10]. We call this a *compatible resistance form*.

Further if μ is a σ-finite Borel measure on X, then $\mathcal{F} \cap L^2(X, \mu)$ is complete under the $\mathcal{E}_1^{1/2}$-norm, so that $(\mathcal{E}, \mathcal{F} \cap L^2(X, \mu))$ is a Dirichlet form on $L^2(X, \mu)$, where

$$\mathcal{E}_1(u) = \mathcal{E}(u) + \int_X u^2 d\mu, \qquad u \in \mathcal{F} \cap L^2(X, \mu).$$

Next, let $\{F_i\}_{i=1}^N$ be an iterated function system (IFS) on $\mathbb{R}^d$ such that

$$F_i(x) = \varrho(x - b_i) + b_i, \quad 1 \le i \le N, \tag{2.5}$$

where $0 < \varrho < 1$ and $b_i \in \mathbb{R}^d$. Let $K = \bigcup_{i=1}^N F_i(K)$ be the corresponding self-similar set, and let μ be the self-similar measure on K defined by $\mu = \frac{1}{N}\sum_{i=1}^N \mu \circ F_i^{-1}$. If the IFS satisfies the open set condition (OSC), i.e. there is a nonempty bounded open set O such that $F_i(O) \subset O$ and $F_i(O) \cap F_j(O) = \emptyset$ for $i \neq j$, then the Hausdorff dimension of K is $\dim_H(K) = \alpha = \frac{\log N}{|\log \varrho|}$, and μ is the α-dimensional Hausdorff measure normalized on K, which is α-regular in the sense that

$$\mu(B(x, r)) \asymp r^\alpha,$$

for $0 < r < \mathrm{diam}(K)$, and $x \in K$ with $B(x, r) := \{y \in K : \ |x - y| < r\}$.

We always assume that K is connected. We define the symbolic space of K as usual. Let $\Sigma = \{1, \cdots, N\}$ be the alphabet, Σ_n be the set of all the words with length n, and Σ_∞ be the set of infinite words $\omega = \omega_1\omega_2 \cdots$; let $\pi : \Sigma_\infty \to K$ be defined by $\{x\} = \{\pi(\omega)\} = \bigcap_{n \ge 1} K_{\omega_1 \cdots \omega_n}$, a symbolic representation of $x \in K$ by ω, where $K_{\omega_1 \cdots \omega_n} = F_{\omega_1} \circ \cdots \circ F_{\omega_n}(K)$.

Following Kigami [21], we define the *critical set* C and the *post-critical set* $\mathcal{P}$ for K by

$$C = \pi^{-1}\Big(\bigcup\nolimits_{1 \le i < j \le N}(K_i \cap K_j)\Big), \quad \mathcal{P} = \bigcup\nolimits_{m \ge 1} \tau^m(C),$$

where $K_i = F_i(K)$, $\tau : \Sigma_\infty \to \Sigma_\infty$ is the left shift by one index. If $\mathcal{P}$ is a finite set, we call $\{F_i\}_{i=1}^N$ a *post-critically finite* (p.c.f.) IFS, and K a p.c.f. (self-similar) set. The *boundary* of K is defined to be $V_0 = \pi(\mathcal{P})$. (We always assume $\#(V_0) \ge 2$ to avoid triviality.) We use $(K, \{F_i\}_{i=1}^N, V_0)$ to denote a p.c.f. triple.

We also define

$$V_n = \bigcup_{i \in \{1, \cdots, N\}} F_i(V_{n-1}), \quad V_* = \bigcup_{n \ge 1} V_n.$$

It is clear that $\{V_n\}_{n=0}^\infty$ is an increasing sequence of sets, and K is the closure of V_*. For any $\omega \in \Sigma_n$, we call $K_\omega := F_\omega(K)$ a *cell* of K, where $F_\omega = F_{\omega_1} \circ \cdots \circ F_{\omega_n}$.

It is known that a p.c.f. IFS in (2.5) satisfies the OSC [6]. (More generally, this is true if the associated similarity matrices A_i of F_i (instead of the ϱ in (2.5)) are commensurable, i.e. there exists a matrix A and integers n_i such that $A_i = A^{n_i}$; but it is claimed in [26] that it is not true without this assumption). Hence the p.c.f. self-similar set K in (2.5) has dimension α, and is associated with a self-similar measure μ that is α-regular.

The following was proved in [9]:

Proposition 2.1. *A p.c.f. IFS in* (2.5) *satisfies the following separation property:*

(H)*: there exists $\delta > 0$ such that for any integer $n \ge 1$ and any two words ω and τ with length n and $K_\omega \cap K_\tau = \emptyset$,*

$$\mathrm{dist}(K_\omega, K_\tau) \ge \delta\varrho^n.$$

We recall that on p.c.f. sets, a regular harmonic structure induces a compatible network sequence which converges to a resistance form satisfying the energy self-similar identity

$$\mathcal{E}(u) = \sum_{i=1}^{N} \frac{1}{r_i} \mathcal{E}(u \circ F_i), \tag{2.6}$$

with $0 < r_i < 1$ for $1 \le i \le N$. In [22], Kigami and Lapidus gave a detailed study of the spectral asymptotic estimate of Laplacians of Dirichlet forms satisfying the energy self-similar identity, and with self-similar measures. Let $\rho(t)$ be the eigenvalue counting function of the Laplacian on K, with Dirichlet or Neumann boundary conditions. They proved [22, Theorem 2.4]

$$0 < \liminf_{t\to\infty} \rho(t) t^{-\frac{d_S}{2}} \le \limsup_{t\to\infty} \rho(t) t^{-\frac{d_S}{2}} < \infty,$$

where d_S is the unique positive number satisfying $\sum_{i=1}^{N} (r_i \mu_i)^{\frac{d_S}{2}} = 1$, and μ_i is the weight of the self-similar measure. Also they gave a refinement of this limit expression using Feller's renewal formula.

3. Homogeneous resistance forms

Let $(K, \{F_i\}_{i=1}^N, V_0)$ be a p.c.f. triple with IFS defined by (2.5), and consider the compatible resistance forms $\{(V_n, E_n)\}_{n=0}^\infty$ with

$$E_n(u) = \sum_{\omega\in\Sigma_n} \sum_{p,q\in V_0} c(F_\omega(p), F_\omega(q)) \Big(u(F_\omega(p)) - u(F_\omega(q))\Big)^2. \tag{3.1}$$

We assume that V_0 is *irreducible* in the sense that for any $p, q \in V_0$, there is a chain $p = p_1, \cdots, p_m = q$ in V_0 such that $c(p_i, p_{i+1}) > 0$ for $i = 1, \cdots, m-1$.

We impose the following two conditions on the compatible sequence $\{(V_n, E_n)\}_n$:

(A1): *For any $n \ge 0$, and any $p, q \in V_0$, $c(F_\omega(p), F_\omega(q))$ is independent of $\omega \in \Sigma_n$, that is*

$$c(F_\omega(p), F_\omega(q)) = c_n(p, q).$$

(A2): *For any distinct $p, q \in V_0$, either $c_n(p, q) = 0$ for all $n \ge 0$, or there exists a constant $\kappa(p, q) > 1$ such that $c_n(p, q) \asymp \kappa(p, q)^n$, i.e. there exists $C > 0$ such that $C^{-1}\kappa(p, q)^n \le c_n(p, q) \le C\kappa(p, q)^n$ for all $n \ge 0$.*

We remark that (A1) implies the conductances are *homogeneous* (or translation invariant) in the sense that all the cells on the same level have equal conductance (Note: the term here is different from the "homogenization" in [13], which is referred to the convergence of a sequence of rescalings without full symmetry group on the natural BM on the fractal); (A2) implies the $c_n(p, q)$ are asymptotically of the form $\kappa(p, q)^n$ depend on p, q, which allows anisotropic growth rate of the conductances. This condition ensures that the topology given by the effective resistance in the limit is equivalent to the topology by the Euclidean metric (see Theorem 3.2). The assumption that $\kappa(p, q) > 1$ can be defined slightly more generally: for any $p, q \in V_0$, there exist a chain $p = p_1, p_2 \cdots, p_{k-1}, p_k = q$ with $\kappa(p_i, p_{i+1}) > 1$, and $c_n(p_i, p_{i+1}) \asymp \kappa(p_i, p_{i+1})^n$, $i = 1, \cdots, k-1$.

The most celebrated examples that satisfy (A1) and (A2) are the Sierpinski gasket and the Vicsek cross with regular harmonic structure, and $c_n(p, q) = \kappa^n$ for some $\kappa > 1$, and for all $p \neq q$ in V_0; in this case $r_1 = \cdots = r_N = \kappa^{-1}$ is the renormalization factor of the self-similar energy identity. Note that the regular harmonic structure in (2.6) with distinct r_i's will not satisfy (A1) as the cells in each level do not have equal conductances. Supplementing to the above examples, recently, two of the authors constructed [10, Theorem 6.1] on an "asymmetric" p.c.f. set (the eyebolted Vicsek cross) a homogeneous resistance form without self-similar energy identity. In that case, conditions (A1) and (A2) are satisfied (see Theorem 5.1), with $c_n(p,q) \asymp \kappa(p,q)^n$, and $\kappa(p,q)$ are distinct among different choices of p, q.

We define $E_n^{(\ell)}(u)$ to be the form in (3.1) with conductances $c_{\ell+n}(p,q)$, a shift of ℓ-level of the conductances. Let $\mathcal{E}^{(0)}(u) = \mathcal{E}(u)$ as in (2.4), and for $\ell > 0$, define $\mathcal{E}^{(\ell)}(u)$ to be the limit of $E_n^{(\ell)}(u)$ as $n \to \infty$. The following is an analogue of the energy self-similar identity.

Proposition 3.1. *On a p.c.f. set, let $(\mathcal{E}, \mathcal{F})$ be a resistance form on V_* defined by* (2.4), (3.1) *and satisfies (A1). Then we have*

$$\mathcal{E}^{(0)}(u) = \sum\nolimits_{i=1}^{N} \mathcal{E}^{(1)}(u \circ F_i), \qquad u \in \mathcal{F} \tag{3.2}$$

and inductively $\mathcal{E}(u) = \sum_{\omega\in\Sigma_\ell} \mathcal{E}^{(\ell)}(u \circ F_\omega)$.

Proof. It follows directly from definition that

$$\sum_{i=1}^{N} E_n^{(1)}(u \circ F_i) = \sum_{i=1}^{N} \sum_{\omega\in\Sigma_n} \sum_{p,q\in V_0} c(F_{i\omega}(p), F_{i\omega}(q))\Big(u(F_{i\omega}(p)) - u(F_{i\omega}(q))\Big)^2 = E_{n+1}(u),$$

and (3.2) follows by taking limit on n. □

We define

$$\kappa_0 = \max\left\{ s : \begin{array}{c} \forall p \neq q \text{ in } V_0, \exists \text{ a chain } p = p_1, p_2, \cdots, p_m = q \text{ in } V_0 \\ \ni \kappa(p_i, p_{i+1}) \geq s, \ \forall\ 1 \leq i \leq m-1 \end{array} \right\} (> 1)\ . \tag{3.3}$$

It is seen that $\kappa_0 \geq \kappa_{\min} := \min\{\kappa(p,q) : p, q \in V_0, p \neq q\}$.

As in Section 2, we let $R(\cdot,\cdot)$ be the corresponding resistance metric on V_*, then R can be extended to the completion $\bar{V}_*$ with respect to R. In the following we see that $\bar{V}_* = K$. Let $\kappa_{\max} := \max\{\kappa(p,q) : p, q \in V_0\}$.

Theorem 3.2. *Suppose $(K, \{F_i\}_{i=1}^N, V_0)$ is a p.c.f. triple. Let $(\mathcal{E}, \mathcal{F})$ be a compatible resistance form on V_* that satisfies (A1) and (A2), and let $R(\cdot,\cdot)$ be the resistance metric on V_*, then R satisfies the estimate*

$$C^{-1}|x-y|^{\frac{\log\kappa_{\max}}{|\log\varrho|}} \leq R(x,y) \leq C|x-y|^{\frac{\log\kappa_0}{|\log\varrho|}}, \quad \forall x, y \in V_*. \tag{3.4}$$

Consequently, the R-closure of V_* is K, and $\mathcal{F} \subset C(K)$, the space of continuous functions on K with respect to R or the Euclidean distance.

Proof. For distinct points x, y in V_*, let ℓ be the integer such that $\delta\varrho^{\ell+1} \le |x-y| < \delta\varrho^{\ell}$, where $\delta > 0$ is a constant in property (H) in Proposition 2.1. We may assume that $\ell \ge 0$. Let ω and τ be two words with length ℓ such that $x \in K_\omega$ and $y \in K_\tau$. Then by (H), we have

$$K_\omega \cap K_\tau \neq \emptyset.$$

To estimate the upper bound of (3.4), we pick $z \in F_\omega(V_0) \cap F_\tau(V_0) \subseteq K_\omega \cap K_\tau$ (it is possible that $\omega = \tau$), and estimate $R(x, z)$. We can find a decreasing sequence of cells $\{K_{\omega_k}\}_{k\ge 0}$ such that $\omega_0 = \omega$, $\omega_k \in \Sigma_{\ell+k}$, $k \ge 0$, and $x \in \bigcap_{k\ge 0} K_{\omega_k}$. Choose an infinite chain $z = z_0, z_1, \cdots$ such that $\lim_{k\to\infty} z_k = x$ and z_k and z_{k+1} are contained in $F_{\omega_k}(V_0)$, $k \ge 0$.

Now for $u \in \mathcal{F}$, $\mathcal{E}(u) \neq 0$, consider $|u(z_k) - u(z_{k+1})|$. Note that $F^{-1}_{\omega_k}(z_k), F^{-1}_{\omega_k}(z_{k+1})$ are in V_0, and thus we can find a chain $F^{-1}_{\omega_k}(z_k) = x_1, \cdots, x_m = F^{-1}_{\omega_k}(z_{k+1})$ in V_0 such that $\kappa(x_j, x_{j+1}) \ge \kappa_0$. Hence

$$\begin{aligned}
|u(z_k) - u(z_{k+1})| &\le \sum_{j=1}^{m-1} |u(F_{\omega_k}(x_j)) - u(F_{\omega_k}(x_{j+1}))| \\
&\le C_1 \max_{\kappa(p,q)\ge\kappa_0} \left\{c_0(p,q)^{-1/2}\right\} \cdot \sum_{j=1}^{m-1} \left(\kappa_0^{-(\ell+k)} \mathcal{E}(u)\right)^{1/2} \\
&\le mC_1 \max_{\kappa(p,q)\ge\kappa_0} \left\{c_0(p,q)^{-1/2}\right\} \cdot \left(\kappa_0^{-(\ell+k)} \mathcal{E}(u)\right)^{1/2}
\end{aligned}$$

(Note that m is uniformly bounded, and C_1 is a constant independent of z_k, z_{k+1} and the chain connecting them). Thus

$$\begin{aligned}
|u(x) - u(z)| \le \sum_{k\ge 0} |u(z_k) - u(z_{k+1})| &\le C_2 \max_{\kappa(p,q)\ge\kappa_0} \left\{c_0(p,q)^{-1/2}\right\} \cdot \sum_{k\ge 0} \left(\kappa_0^{-(\ell+k)} \mathcal{E}(u)\right)^{1/2} \\
&\le C_3 \max_{\kappa(p,q)\ge\kappa_0} \left\{c_0(p,q)^{-1/2}\right\} \cdot \kappa_0^{-\ell/2} \mathcal{E}(u)^{1/2}. \qquad (3.5)
\end{aligned}$$

It follows from (2.1) that

$$R(x, z) \le C_4 \max_{\kappa(p,q)\ge\kappa_0} \left\{c_0(p,q)^{-1}\right\} \cdot \kappa_0^{-\ell}. \qquad (3.6)$$

Similarly, we have the same bound for $R(y, z)$. Hence

$$R(x, y) \le R(x, z) + R(y, z) \le C_5 \kappa_0^{-\ell} \le C|x-y|^{-\frac{\log\kappa_0}{\log\varrho}}.$$

For the lower bound of (3.4), we can find a positive integer m_0, and an $(\ell+m_0)$-cell $K_{\widetilde{\omega}}$ containing x, such that $K_{\widetilde{\omega}}$ does not intersect any $(\ell+m_0)$-cell containing y. Then let h be the $(\ell+m_0)$-piecewise harmonic function with values 1 on $K_{\widetilde{\omega}}$, and 0 on any other points in $V_{\ell+m_0}$. Hence $h(y) = 0$ and we have

$$\mathcal{E}(h) \le C_6 \max_{p,q\in V_0} c_{\ell+m_0}(p,q) \le C_7 \kappa_{\max}^{\ell},$$

where C_6 and C_7 are constants independent of ℓ and x, y. Hence by (2.2), we have

$$R(x, y) \ge \mathcal{E}(h)^{-1} \ge C_7^{-1} \kappa_{\max}^{-\ell} \ge C^{-1}|x-y|^{-\frac{\log\kappa_{\max}}{\log\varrho}}.$$

That the closure of V_* equals K follows from the expression in (3.4). For $u \in \mathcal{F}$, $|u(x) - u(y)|^2 \le R(x, y)\mathcal{E}(u)$, $x, y \in K$. This implies u is continuous with respect to R, and also to the Euclidean distance by (3.4). □

Corollary 3.3. *With the assumptions in Theorem 3.2, let μ be a σ-finite Borel measure on (K, R) will full support. Then $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(K, \mu)$.*

Proof. The regularity of $(\mathcal{E}, \mathcal{F})$ can be seen by using piecewise harmonic functions to approximate any $u \in C(K)$, and hence $\mathcal{F}$ is dense in $C(K)(\subset L^2(K, \mu))$. For the locality, suppose $u, v \in \mathcal{F}$ have disjoint supports. Then $u|_{V_n}, v|_{V_n}$ have disjoint support for n large enough. An application of (3.1) shows that $E_m(u, v) = 0$ for $m \geq n$. By taking limit on m, $\mathcal{E}(u, v) = 0$. □

Definition 3.4. *On a p.c.f. self-similar set K, we call $(\mathcal{E}, \mathcal{F})$ a* homogeneous *resistance form (RF) if it is defined by a compatible resistance form satisfying (A1) and (A2) as in Theorem 3.2, and call it* degenerated homogenous *RF if it does not satisfy the energy self-similar identity.*

The corresponding terminologies are used for the local regular Dirichlet form associated with a measure μ, and Δ_μ is used to denote the induced Laplacian.

We need the following estimation in the next section.

Lemma 3.5. *Let $(\mathcal{E}, \mathcal{F})$ be a homogeneous resistance form on a p.c.f. set K, and let $R^{(n)}$ be the resistance metric of $(\mathcal{E}^{(n)}, \mathcal{F})$. Then $R^{(n)}$ is bounded on K and satisfies*

$$R^{(n)}(x, y) \leq C\kappa_0^{-n}, \qquad x, y \in K, \tag{3.7}$$

where C is independent of $n \geq 0$.

Proof. We first consider $n = 0$. For $x \in K$, let $B(x, \delta)$ be an open ball of radius δ (in the Euclidean distance). Then referring to the proof of Theorem 3.2 for $\ell = 0$, and by (3.6), we have

$$R(x, y) \leq C_4 \max_{\kappa(p,q) \geq \kappa_0} \left\{ c_0(p, q)^{-1} \right\}, \qquad x, y \in B(x, \delta) \cap K. \tag{3.8}$$

As K is compact, we can cover it by finitely many open balls $B(x, \delta)$'s. This implies

$$R(x, y) \leq C, \qquad x, y \in K \tag{3.9}$$

for some $C > 0$.

For $\mathcal{E}^{(n)}, n > 0$, the conductances on V_0 are $c_n(p, q)$ and on V_k are $c_{n+k}(p, q)$. Similar to (3.8), we obtain

$$R^{(n)}(x, y) \leq C_4 \max_{\kappa(p,q) \geq \kappa_0} \left\{ c_n(p, q)^{-1} \right\} \leq C\kappa_0^{-n}, \qquad x, y \in K$$

for the same $C > 0$. □

4. Spectral asymptotics

In this section we will consider the homogeneous Dirichlet form associated with a measure μ and a Laplacian $\Delta = \Delta_\mu$. We first introduce one more assumption.

Let κ_0 be as in (3.3). For $p, q \in V_0$, we define $p \sim_0 q$ if

$$\text{either } p = q \quad \text{or} \quad \exists \text{ a chain } p = p_1, \cdots, p_m = q \ni \kappa(p_i, p_{i+1}) > \kappa_0.$$

This is an equivalence relation on V_0. We also consider the induced equivalence relation $\sim_1$ on V_1: the minimal equivalence relation on V_1 such that for any $k \in \Sigma$ and any $p \sim_0 q$, $F_k(p) \sim_1 F_k(q)$ (i.e. for $p, q \in V_1$, either $p = q$; or there exists a chain $p = p_1, \cdots, p_m = q$ in V_1 joining p, q, and for each i, there exists $k \in \Sigma$ such that $F_k^{-1}(p_i) \sim_0 F_k^{-1}(p_{i+1})$). The same way, we can define $\sim_n$ on V_n.

Let $C_i^{(0)}$ denote a nontrivial (non-singleton) connected component induced by $\sim_0$ (i.e. a non-trivial equivalent class in V_0), and let $\mathfrak{C}^{(0)}$ be the family of $C_i^{(0)}$'s. Similarly we let $C_j^{(n)}$, $\mathfrak{C}^{(n)}$ be the connect components and associated family of connected components of $\sim_n$ in V_n. The following are some simple observations.

(i) If $\mathfrak{C}^{(0)} = \emptyset$, then all the connected components are singletons and $\kappa_0 = \kappa_{\max}$; if $\mathfrak{C}^{(0)} \neq \emptyset$, then $C_i^{(0)} \subset V_0 \setminus \{p\}$ for some $p \in V_0$, following from the definition of κ_0;

(ii) For each map F_k in the IFS, $F_k(C_i^{(0)}) \subset C_j^{(1)}$ for some j. In general, equality does not hold, as $C_j^{(1)}$ can be a union of several $F_{k'}(C_{i'}^{(0)})$'s through the junction vertices of these cells.

We define an assumption on the homogeneous resistance form:

(A3) *there exists n_0 and an n_0-cell K_ω such that for any $p \in F_\omega(V_0)$, $q \in V_0$, $p \nsim_{n_0} q$.*

It is clear that if $\mathfrak{C}^{(0)} = \emptyset$, then (A3) is trivially satisfied for n_0 that has an n_0-cell not intersecting V_0 (by (i)). Also

Lemma 4.1. *If (A3) is satisfied at n_0, then (A3) is also satisfied for $n > n_0$.*

Proof. Suppose (A3) holds at n_0. Let K_ω be the n_0-cell satisfying the condition in (A3), and pick arbitrarily an $i \in \Sigma$. Then the $(n_0 + 1)$-cell $K_{i\omega}$ will satisfy (A3) as well.

By repeating this argument to $n_0 + 1$, we have (A3) holds for $n_0 + 2$. Inductively, we have (A3) holds for any $n > n_0$. □

The following are some simple examples.

Example 1. (Sierpinski triangle). Let $\{p_1, p_2, p_3\}$ be the three vertices of a Sierpinski triangle, and let the nontrivial component be $\{p_1, p_2\}$ in V_0. Then the components in V_n are line segments in V_n parallel to $\overline{p_1 p_2}$. Clearly $(A3)$ is satisfied with $n_0 = 2$.

Example 2. (Pentagasket) Let $\{p_i\}_{i=1}^5$ be the vertices of the pentagasket. If we let $\{p_1, p_2\}$, $\{p_3, p_4, p_5\}$ be two components in V_0. Then the connected components in V_1, V_2 are as in the left two pictures Figure 1, and $n_0 = 2$.

If we take $\{p_1, p_2, p_3, p_4\}$ as the component in V_0, then the connected components in V_1, V_2 are as in the right two pictures in Figure 1, and $n_0 = 2$.

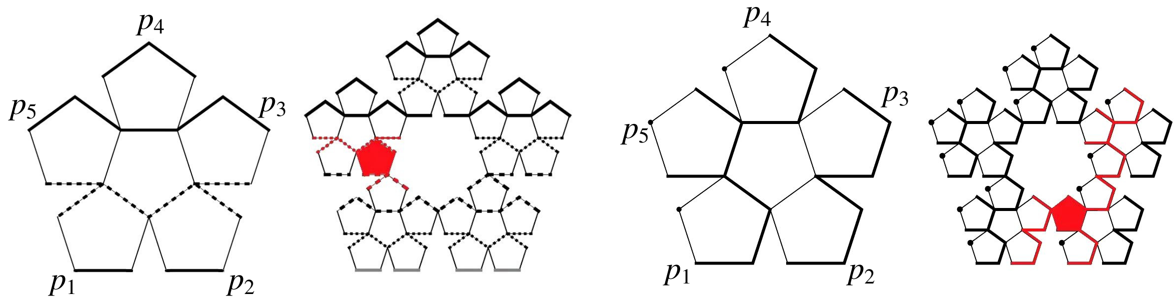

FIGURE 1. The connected components of V_1 and V_2 in boldface and dotted line segments, and dots; the cell in red satisfies (A3).

Example 3. (Snowflake) Let $V_0 = \{p_i\}_{i=1}^6$ be the vertices of a hexagon. Let $\{F_i\}_{i=1}^7$ be as in Figure 2. Consider the component $\{p_i\}_{i=1}^5$ in V_0. Then there is only one non-trivial component in V_1, which meets all vertices in V_0 except one. By similarity, it is seen that it is the same for V_n. Hence (A3) is not satisfied.

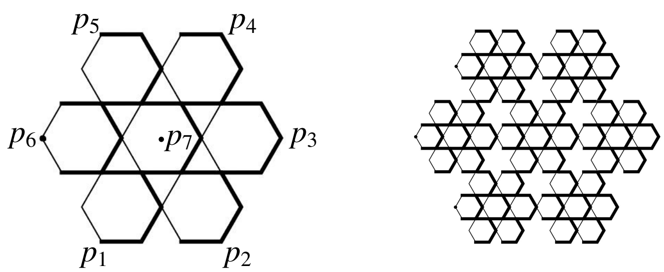

FIGURE 2. The connected components of V_1 and V_2 in boldface segments and dots; the assumption (A3) fails.

In view of Theorem 3.2, it is known that under the $\mathcal{E}_1^{1/2}$-norm, $\mathcal{F}$ can be compactly embedded into $L^2(K, \mu)$, and $-\Delta$ has a discrete spectrum which only has a limit point ∞ [21, Theorem 2.4.1]. We will consider the first eigenvalue λ_1 of $-\Delta$ with the Dirichlet boundary condition (i.e. on $\mathcal{F}_0 := \{u \in \mathcal{F} : u|_{V_0} = 0\}$). Note that λ_1 can be expressed by the celebrated Rayleigh quotient:

$$\lambda_1 = \inf_{u \in \mathcal{F}_0, u \neq 0} \frac{\mathcal{E}(u)}{\|u\|_2^2}. \tag{4.1}$$

Let $\mathcal{E}^{(0)} = \mathcal{E}$, and let $\mathcal{E}^{(n)}$ be defined as before Proposition 3.1. Let $\lambda_1^{(n)}, n \geq 0$ be the first eigenvalue of $(\mathcal{E}^{(n)}, \mathcal{F})$.

Lemma 4.2. *Let K be a p.c.f. set satisfying (2.5) and μ be the normalized Hausdorff measure on K. Let $(\mathcal{E}, \mathcal{F})$ be a homogeneous Dirichlet form in $L^2(K, \mu)$ satisfying (A3). Then there is $C > 0$ such that for $(\mathcal{E}^{(n)}, \mathcal{F})$, $n \geq 0$, we have*

$$C^{-1}\kappa_0^n \leq \lambda_1^{(n)} \leq C\kappa_0^n. \tag{4.2}$$

Proof. Let $n \geq 0$. By (A3), we can find an integer $n_0 \geq 1$ and an n_0-cell K_ω, such that each component containing a point in $F_\omega(V_0)$ is disjoint from V_0. Denote by Q the union of all such components. Then $F_\omega(V_0) \subseteq Q$ and $Q \cap V_0 = \emptyset$. Let $u_0 \in \ell(V_{n_0})$ by taking 1 on Q, 0 on $V_{n_0} \setminus Q$, and extent harmonically to K. Clearly,

$$\|u_0\|_2^2 \geq \int_{F_\omega(K)} u_0^2 \, d\mu \;=\; \mu(K_\omega) = N^{-n_0}.$$

Define the boundary ∂Q of Q in V_{n_0} to be those $x \in Q$ that has a neighbor $y \in V_{n_0} \setminus Q$ such that $c_{n+n_0}(x, y) > 0$ (recall that the energy $\mathcal{E}^{(n)}$ is defined using the conductance $c_{n+m}(p, q)$ and letting $m \to \infty$); let Q_x denote those y. Note that x and y are in different components, and hence $c_{n+n_0}(x, y) \leq C\kappa_0^{n+n_0}$, where C is independent of n; also note that $\#Q_x$ is uniformly bounded by some $C_0 > 0$, and $\#\partial Q$ is bounded by $(\#(V_0)N)^{n_0}$. Hence by (3.1),

$$\mathcal{E}^{(n)}(u_0) = \sum_{x\in\partial Q, y\in Q_x} c_{n+n_0}(x, y)(u_0(x) - u_0(y))^2 \leq C_0(\#(V_0)N)^{n_0} \cdot C\kappa_0^{n+n_0} =: C_1\kappa_0^n.$$

Therefore

$$\lambda_1^{(n)} \leq \frac{\mathcal{E}^{(n)}(u_0)}{\|u_0\|_2^2} \leq C_1\kappa_0^n \cdot N^{n_0} \leq C_2\kappa_0^n$$

for some $C_2 > 0$.

To estimate the lower bound, we let $u \in \mathcal{F}_0 = \{u \in \mathcal{F} : u|_{V_0} = 0\}$, $u \neq 0$. Then

$$|u(x) - u(y)|^2 \leq R^{(n)}(x, y)\mathcal{E}^{(n)}(u), \quad x, y \in K.$$

By choosing y to be some $p \in V_0$, we have

$$|u(x)|^2 \leq R^{(n)}(x, p)\mathcal{E}^{(n)}(u), \quad \forall x \in K.$$

Integrating both sides with respect to μ, we obtain

$$\|u\|_2^2 \leq \int_K R^{(n)}(x, p)d\mu(x) \cdot \mathcal{E}^{(n)}(u).$$

Using Lemma 3.5, we have $C_3 > 0$ such that

$$C_3\kappa_0^n \leq \frac{\mathcal{E}^{(n)}(u)}{\|u\|_2^2}.$$

Since u is arbitrary, this implies that $C_3\kappa_0^n \leq \lambda_1^{(n)}$. Hence $C^{-1}\kappa_0^n \leq \lambda_1^{(n)} \leq C\kappa_0^n$ for some $C > 0$. □

Denote by $\Lambda_{\mathcal{D}}(\lambda_1)$ the Dirichlet eigensubspace of λ_1. In the following we show that $\dim \Lambda_{\mathcal{D}}(\lambda_1) = 1$ in a general setting. This result is implicitly implied in the monographs of Davies [4, Theorems 7.2, 7.3] and [5, Propostion 1.4.3] in the study of heat kernels and heat semigroups, $\{P_t\}_{t\geq 0}$, i.e. $\{e^{-t\Delta}\}_{t\geq 0}$. For completeness, we will give a modified proof here following his main idea.

We say that a linear subspace $\mathcal{L}$ of L^2 is a *sublattice* if $u \in \mathcal{L}$ implies $|u| \in \mathcal{L}$.

Proposition 4.3. *Let K be a compact connected set and ν a σ-finite Borel measure on K with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(K, \nu)$ with $\mathcal{F} \subset C(K)$. Suppose $\mathcal{L} \subset \mathcal{F}$ is a closed linear sublattice of $L^2(K, \nu)$, and there exists $C > 0$ such that*

$$P_t u \le Cu, \qquad \forall\, t > 0,\;\; u \ge 0,\;\; u \in \mathcal{L}. \tag{4.3}$$

Then $\mathcal{L}$ has dimension at most one.

Proof. Suppose $\mathcal{L}$ is nontrivial. Let $u \ge 0$ be any non-zero element in $\mathcal{L}$. Then $u \in C(K)$. Let $U = \{x \in K :\; u(x) \neq 0\}$. Then U is open. We claim that $U = K$. Indeed, if $\upsilon \in C(K)$ and $|\upsilon| \le \alpha u$ for some $\alpha \ge 0$, then by the Markovian property of $\{P_t\}_{t>0}$ and (4.3), we have

$$|P_t \upsilon| \le P_t |\upsilon| \le \alpha P_t u \le \alpha C u.$$

Hence for

$$\mathcal{G} = \{\upsilon \in C(K) : |\upsilon| \le \alpha u \;\text{ for some } \alpha \ge 0\},$$

$P_t(\mathcal{G}) \subseteq \mathcal{G}$ for all $t \ge 0$. As U is an open set, $\mathcal{G}$ contains all the continuous functions that are compactly supported in U. Then the L^2-closure of $\mathcal{G}$ is the set of all $\upsilon \in L^2(K, \nu)$ with $\upsilon = 0$ on $K \setminus U$. So U is an *invariant* set of the semigroup $\{P_t\}_{t>0}$. (A ν-measurable set $B \subset K$ is said to be P_t-invariant if $P_t(1_B f) = 1_B P_t f$ ν-a.e. for any $f \in L^2$ and $t > 0$.) This implies $1_U = K$ ν-a.e. [4, Theorems 7.2]. On the other hand, by [8, Theorem 1.6.1], $1_U \in \mathcal{F}$, which is contained in $C(K)$ by assumption. Since K is connected, we have $U = K$ or $U = \emptyset$. Since u is nonzero, we conclude that $U = K$, and the claim follows.

Now, if $u \in \mathcal{L}$, then u^+ and u^- are in $\mathcal{L}$ and have disjoint supports. It follows from the claim that one of them must vanish. Hence $u \in \mathcal{L}$ implies $u \ge 0$ or $-u \ge 0$. If u, u' are two distinct positive elements of $\mathcal{L}$, then $u + \eta u'$ is either positive or negative for all $\eta \in \mathbb{R}$. But the sum must change sign as η increases through $\mathbb{R}$. Hence there is η such that $u + \eta u' = 0$. Hence $\mathcal{L}$ is one dimensional. □

Proposition 4.4. *With the same assumption as Proposition 4.3,* $\dim \Lambda_{\mathcal{D}}(\lambda_1) = 1$.

Proof. We make use of the Rayleigh quotient again. Let $u \in \mathcal{F}$ attain the infimum in (4.1), and all such functions must be eigenfunctions of $\mathcal{D}$-eigenvalue λ_1.

We show that $\Lambda_1 := \Lambda_{\mathcal{D}}(\lambda_1)$ is a closed sublattice. Indeed for any nonzero $u \in \Lambda_1 \subseteq \mathcal{F}_0$, we see by the Markovian property of the Dirichlet form that $|u| \in \mathcal{F}_0$, and $\mathcal{E}(|u|) = \mathcal{E}(u)$ (use the equivalence of Markovian property and the normal contraction condition in [8, Theorem 1.4.1]). Hence $|u|$ attains the infimum in (4.1), and is also an eigenfunction of λ_1. This shows that Λ_1 is a linear sublattice.

For the closedness of Λ_1, we claim that

$$\Lambda_1 = \{u \in L^2 :\; P_t u = e^{-t\lambda_1} u \text{ for all } t \ge 0\}\,. \tag{4.4}$$

Then Λ_1 will be closed. To see the claim, on the one hand, for any $u \in \Lambda_1$, we have $u \in \operatorname{dom} \Delta$ and $\Delta u = -\lambda_1 u$. Hence

$$P_t u = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Delta^n u = \sum_{n=0}^{\infty} \frac{t^n}{n!} (-\lambda_1)^n\, u = e^{-t\lambda_1} u\,. \tag{4.5}$$

On the other hand, for $u \in L^2$ satisfying the right hand side of (4.4), then the following limit exists

$$\lim_{t \to 0+} \frac{P_t u - u}{t} = -\lambda_1 u\,,$$

which implies that $u \in \operatorname{dom} \Delta$ and $\Delta u = -\lambda_1 u$ and thus $u \in \Lambda_1$. The claim is proved.

By (4.5), we see that $P_t u \leq Cu$. Then Proposition 4.3 implies Λ_1 is of dimension one. □

For the discrete eigenvalues of $-\Delta$, we use $\rho_b(t)$ to denote the b-eigenvalue counting function of $-\Delta$, where b stands for $\mathcal{D}$ or $\mathcal{N}$, the Dirichlet or Neumann boundary condition:

$$\rho_b(t) = \sum_{\lambda \leq t} \dim \Lambda_b(\lambda),$$

where $\dim \Lambda_b(\lambda)$ is the dimension of the λ-eigensubspace. We use $\rho_b^{(n)}(t)$ to denote the b-eigenvalue counting function corresponding to $(\mathcal{E}^n, \mathcal{F})$.

Lemma 4.5. *Let K be a p.c.f. self-similar set satisfying (2.5), and let μ be the normalized Hausdorff measure on K. Suppose $(\mathcal{E}, \mathcal{F})$ is a homogeneous Dirichlet form in $L^2(K, \mu)$. Then for each $(\mathcal{E}^{(n)}, \mathcal{F}), n \geq 0$, we have*

$$N^k \rho_{\mathcal{D}}^{(n)}\left(\frac{t}{N^k}\right) \leq \rho_{\mathcal{D}}^{(0)}(t), \quad \textit{and} \quad \rho_{\mathcal{N}}^{(0)}(t) \leq N^k \rho_{\mathcal{N}}^{(n)}\left(\frac{t}{N^k}\right) \tag{4.6}$$

for all $t \geq 0$ and $k \geq 0$.

Proof. The main idea of proof is based on [22, Propsitions 6.2, 6.3]. Let $\mathcal{F}_1 := \{u \in \mathcal{F} : u|_{V_1} = 0\}$. Denote by $\rho_{\mathcal{D}}\left(t; \mathcal{E}^{(0)}, \mathcal{F}_1\right)$ the corresponding eigenvalue counting function. We first show that

$$\rho_{\mathcal{D}}\left(t; \mathcal{E}^{(0)}, \mathcal{F}_1\right) = N \rho_{\mathcal{D}}^{(1)}\left(\frac{t}{N}\right). \tag{4.7}$$

Let λ be a $\mathcal{D}$-eigenvalue of $\mathcal{E}^{(0)}$ with eigenvector $\varphi \in \mathcal{F}_1$. Let $\varphi_i = \varphi \circ F_i$ on K, then for $g \in \mathcal{F}_1$, write $g_i = g \circ F_i \in \mathcal{F}_0 (= \{u \in \mathcal{F} : u|_{V_0} = 0\})$. By Proposition 3.1,

$$\begin{aligned}
&\sum_{i=1}^N \mathcal{E}^{(1)}(\varphi_i, g_i) = \mathcal{E}^{(0)}(\varphi, g) = \lambda(\varphi, g)_{L^2} = \lambda \sum_{i=1}^N \int_{K_i} \varphi g \, d\mu \\
=&\lambda \sum_{i=1}^N \int_{K_i} (\varphi_i \circ F_i^{-1})(g_i \circ F_i^{-1}) \, d\mu = \lambda \sum_{i=1}^N \int_K \varphi_i g_i \, d\mu \circ F_i = \frac{\lambda}{N} \sum_{i=1}^N \int_K \varphi_i g_i \, d\mu.
\end{aligned}$$

This implies $\mathcal{E}^{(1)}(\varphi_i, g) = \frac{\lambda}{N}(\varphi_i, g)$ for all $i \in \Sigma$ and $g \in \mathcal{F}_0$. It follows that φ_i is a $\frac{\lambda}{N}$-eigenvalue of $\mathcal{E}^{(1)}$. Also $\frac{\lambda}{N}$ has multiplicity N with respect to $(\mathcal{E}^{(1)}, \mathcal{F}_0)$.

Similarly, let ψ be a λ-eigenvector of $(\mathcal{E}^{(1)}, \mathcal{F}_0)$, we can show that $\psi_i := (\psi \circ F_i^{-1}) \cdot \chi_{K_i}$ is an $N\lambda$-eigenvector of $(\mathcal{E}^{(0)}, \mathcal{F}_1)$, and $N\lambda$ has multiplicity N in $(\mathcal{E}^{(0)}, \mathcal{F}_1)$.

Combining the above proofs, we obtain (4.7). Using Proposition 3.1 on $\mathcal{E}^{(n)}$ and the same proof, we can show that

$$\rho_{\mathcal{D}}\left(t; \mathcal{E}^{(n)}, \mathcal{F}_1\right) = N \rho_{\mathcal{D}}^{(n+1)}\left(\frac{t}{N}\right). \tag{4.8}$$

Note that for all $n \geq 0$, we have $\rho_{\mathcal{D}}\left(t; \mathcal{E}^{(n)}, \mathcal{F}_1\right) \leq \rho_{\mathcal{D}}^{(n)}(t)$, and together with (4.7), the first inequality in (4.6) follows.

For the Neumann eigenvalues, we consider the domain

$$\mathcal{F}_2 = \{u \in \ell(K \setminus V_1) : u \circ F_i := u_i \text{ on } K \setminus V_0 \text{ for some } \; u_i \in \mathcal{F}\} \; (\supset \mathcal{F}),$$

(see [22, Propositions 5.1, 6.2]). By applying a similar proof as the above, we can show that

$$\rho_{\mathcal{N}}\left(t;\mathcal{E}^{(n)},\mathcal{F}_2\right)=N\rho_{\mathcal{N}}^{(n+1)}\left(\frac{t}{N}\right),$$

and observe that $\rho_{\mathcal{N}}\left(t;\mathcal{E}^{(n)},\mathcal{F}_2\right)\geq\rho_{\mathcal{N}}^{(n)}(t)$, the second inequality in (4.6) follows. □

Theorem 4.6. *Let K be a p.c.f. self-similar set satisfying (2.5) and μ be the normalized Hausdorff measure on K. Let $(\mathcal{E},\mathcal{F})$ be a homogeneous Dirichlet form on $L^2(K,\mu)$ satisfying (A3). Then for $t_0=\inf\{t:\rho(t)>0\}$,*

$$\rho_{\mathcal{D}}(t)\ \asymp\ t^{\frac{\log N}{\log(N\kappa_0)}},\qquad t>t_0.$$

Similarly, the same inequality holds when $\rho_{\mathcal{D}}(t)$ is replaced by $\rho_{\mathcal{N}}(t)$ and for any $t_0>0$.

Proof. We denote by $\lambda_1^{(n)}$ the first $\mathcal{D}$-eigenvalue of the Laplacian associated with $\mathcal{E}^{(n)}$. Then by Proposition 4.4, we have $\rho_{\mathcal{D}}^{(n)}\left(\lambda_1^{(n)}\right)=1$, and $\rho_{\mathcal{N}}^{(n)}\left(\lambda_1^{(n)}\right)\leq\rho_{\mathcal{D}}^{(n)}\left(\lambda_1^{(n)}\right)+N=N+1$ [22, Lemma 2.3]. Then by Lemma 4.5, we have

$$\rho_{\mathcal{N}}^{(0)}\left(N^n\lambda_1^{(n)}\right)\leq C\cdot N^n.$$

Letting $t=N^n\lambda_1^{(n)}$, by Lemma 4.2, we have $t\asymp N^n\kappa_0^n$, hence $N^n\asymp t^{\log N/\log(N\kappa_0)}$. It follows that

$$\rho_{\mathcal{D}}(t)=\rho_{\mathcal{D}}^{(0)}(t)\leq\rho_{\mathcal{N}}^{(0)}(t)\leq Ct^{\frac{\log N}{\log(N\kappa_0)}}$$

for some $C>0$. The same argument yields the other inequality. □

5. Construction of homogeneous forms

In this section, we consider the construction of the homogeneous resistance forms. Let G_n denote the resistance network on V_n. The basic idea is the following: we use $\mathbf{x}_n$ to denote the resistance (or conductance) on the edges of G_n under the homogeneous assumption (A1). Then the compatibility condition implies the existence of a map Φ (the trace map) such that $\Phi(\mathbf{x}_{n+1})=\mathbf{x}_n, n\geq 0$. Inductively we have $\Phi^n(\mathbf{x}_n)=\mathbf{x}_0$. If Φ is invertible, then we have

$$\Phi^{-n}(\mathbf{x}_0)=\mathbf{x}_n,\quad n\geq 1. \tag{5.1}$$

In other word, for a given $\mathbf{x}_0$, we can get a compatible sequence of resistance (or conductance) ([17], [11]). We call this a *reverse recursive construction*.

This simple idea is by no means easy to implement. First to establish the map Φ, it may need to use some network reduction techniques such as resistances in series and in parallel, and the Δ-Y transform as well as its analogs. Secondly, even if the inverse Φ^{-1} exists, it may be impossible to find or too complicated to make it useful. Thirdly, it is not easy to determine the domain of $\mathbf{x}_0$ so that the resulting sequences $\mathbf{x}_n$ exists and satisfies (A2).

In the following, we present two examples, of homogeneous resistance forms does satisfy the energy self-similar identity, to illustrate the above points. The first one is the construction of such a form on the asymmetric eyebolted Vicsek cross [10]; the other one is on a 3-dimensional SG, which is has ”two-parameter conductances”, and the proof is substantially different from the ”two-parameter conductances” in [17, 13].

Example 1. Eyebolted Vicsek cross(EVC)

In $\mathbb{R}^2$, let $\{p_1, p_2, p_3, p_4\}$ be the four vertices of the unit square S as in Figure 3. Divide S into a mesh of sub-squares of size $1/9$, and pick 21 sub-squares as shown in Figure 3.

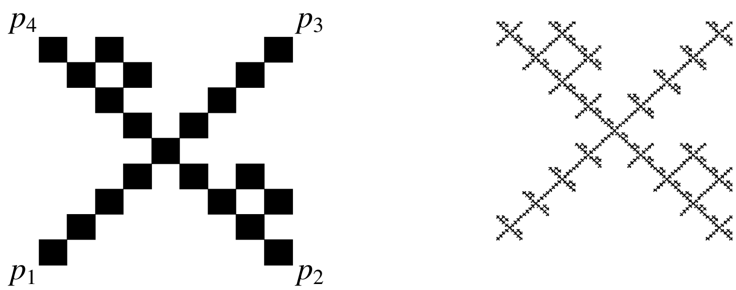

FIGURE 3. The eyebolted Vicsek cross.

For each sub-square Q, let $F_Q : S \to S$ be given by

$$F_Q(x) = x/9 + p_Q$$

where p_Q is chosen so that $F_Q(S) = Q$. Renumber the maps F_Q by $\{F_i\}_{i=1}^{21}$. Let K be the unique nonempty compact set such that $K = \bigcup_{i=1}^{21} F_i(K)$. Then $(K, \{F_i\}_{i=1}^{21})$ is a p.c.f. self-similar set with boundary $V_0 = \{p_1, p_2, p_3, p_4\}$. We call this modified Vicsek cross an *eyebolted Vicsek cross* (EVC). The Hausdorff dimension of K is $\alpha = \log 21/\log 9$, and the self-similar measure with the natural weight is the normalized α-dimensional Hausdorff measure μ on K.

It was shown in [10, Theorem 6.1] that on the eyebolted Vicsek cross, there are two local regular Dirichlet forms that can be constructed. One satisfies the energy self-similar identity (2.6) (but cannot use the Brouwer's fixed-point technique as in [21] since the underlying set is not symmetric); the other one is from the reverse recursive construction and is a degenerated RF. Now we will continue the study of the second construction [10, Theorem 6.1] and show the $\kappa(p, q)$ explicitly.

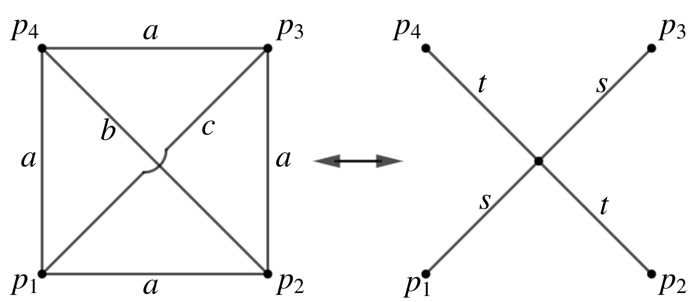

FIGURE 4. the ⊠-X transform of resistance networks.

Analogous to the well-known Δ-Y transform for resistance network with three vertices, in [10, Section 2], we introduced two equivalent networks with four vertices, called the

⊠-X transform, to handle resistance networks composed of aggregate of squares in the self-similar p.c.f sets. The transform is ([10, Lemma 2.7]): assume the resistance on G_0 is $bc = a^2$, then

$$a = 2(s+t), \quad b = \frac{2s}{t}(s+t), \quad \text{and} \quad c = \frac{2t}{s}(s+t) \tag{5.2}$$

(See Figure 4). In the following, for a given initial resistance on G_0 of the EVC (i.e. on the ⊠-side), we transform it to G_0' on the X-side. Then evaluate the compatible resistances on G_n', follow by a transform back to the G_n of EVC.

Theorem 5.1. *On the EVC, suppose we are given equal initial conductance $c_0(p_i, p_j), 1 \leq i, j \leq 4, i \neq j$, on each edge. Then there is a unique homogeneous resistance form with conductances*

$$c_n(p_1, p_3) \asymp 9^n, \quad c_n(p_2, p_4) \asymp \Big(\frac{64}{9}\Big)^n, \ \ \textit{and} \ \ c_n(p_i, p_{i+1}) \asymp 8^n, \ 1 \leq i \leq 4 \ (p_5 = p_1).$$

It is a degenerated resistance form.

Proof. We will consider the corresponding resistance through the ⊠-X transform. Without loss of generality, we can assume the initial resistance $\mathbf{y}_0 = (1, 1, 1, 1)$ on G_0'. Let $\mathbf{y}_n = (s_n, t_n, s_n, t_n)$ be the resistance on the n-level. As in [10, Lamma 5.1 and p.1643], the compatibility condition on the n-th level is $\Phi(\mathbf{y}_n) = \mathbf{y}_{n-1}$, and Φ is calculated to satisfy $\Phi(s, t, s, t) = (9s,\ 9t - \frac{t^2}{s+t},\ 9s,\ 9t - \frac{t^2}{s+t})$. It follows that $\Phi^n(\mathbf{y}_n) = \mathbf{y}_0$, hence $\Phi^{-n}(\mathbf{y}_0) = \mathbf{y}_n$. Solving the above equation for Φ^{-1}, we have

$$\mathbf{y}_n = \Phi^{-1}(\mathbf{y}_{n-1}) = (s_n, t_n, s_n, t_n), \quad \text{with} \ \ s_n = 9^{-n}, \ t_{n-1} = 9t_n - \frac{t_n^2}{9^{-n} + t_n}.$$

We claim that $t_n \asymp 8^{-n}$. For this, we let $x_n = 8^n t_n$, and using the above relationship of t_{n-1} and t_n, we have

$$x_{n-1} = x_n + \frac{1}{8} \cdot \frac{x_n}{1 + (9/8)^n x_n}. \tag{5.3}$$

Observe that $\{x_n\}$ is a non-negative, decreasing sequence, and hence $\lim_{n\to\infty} x_n$ exists. We show that the limit is strictly positive. Then the claim follows.

Fix any $n_0 > 0$, and for $n \geq n_0$, we see by (5.3),

$$\begin{aligned} x_n &= x_{n_0} + \sum_{k=n_0}^{n-1} (x_{k+1} - x_k) \\ &= x_{n_0} - \frac{1}{8} \cdot \sum_{k=n_0}^{n-1} \frac{x_{k+1}}{1 + y_{k+1}} \qquad (\text{letting} \ \ y_k = 9^k t_k) \\ &\geq x_{n_0}\Big(1 - \frac{1}{8} \cdot \sum_{k=n_0}^{\infty} \frac{1}{y_{k+1}}\Big). \end{aligned} \tag{5.4}$$

Again, by checking the relation of t_{n-1} and t_n, we see that $\{y_n\}_n$ satisfies

$$y_{n-1} = y_n - \frac{y_n^2}{9(1 + y_n)}. \tag{5.5}$$

It is an increasing sequence, and $\lim_{n\to\infty} y_n = \infty$ (if otherwise, $\lim\limits_{n\to\infty} y_n = y$ is finite, and by letting $n \to \infty$, we have $y = y - \frac{y^2}{9(1+y)}$, which implies $y = 0$, contradicting that $y_0 = 1$ and y_n is increasing). Hence we have

$$\frac{y_n}{y_{n-1}} = \left(1 - \frac{y_n}{9(1+y_n)}\right)^{-1} > \tau$$

for some $\tau > 1$. This yields $y_n \geq \tau^n$. Substituting this into (5.4), we have

$$x_n \geq \frac{1}{2} x_{n_0},$$

proving that $t_n \asymp 8^{-n}$.

Finally, we transform the resistances s_n and t_n back to G_n; by (5.2), the resistances on G_n satisfies

$$c_n(p_1, p_3)^{-1} = 2(s_n + \frac{s_n^2}{t_n}), \qquad c_n(p_2, p_4)^{-1} = 2(t_n + \frac{t_n^2}{s_n}),$$
$$c_n(p_i, p_{i+1})^{-1} = 2(s_n + t_n),\ i = 1, 2, 3, 4,\ (p_5 = p_1).$$

Substituting $s_n \asymp 9^{-n}$ and $t_n \asymp 8^{-n}$ in the above, we obtain the compatible conductance on G_n as stated.

The last statement was proved in [10, Theorem 6.1]. □

Remark. By checking the compatible conductance in the theorem, it appears that the homogeneous resistance form support an approximate one-dimensional diffusion in the direction of $\overline{p_1 p_3}$, and also diffuse in the other directions with slower rates according to the conductances.

Corollary 5.2. *Let K be the EVC, μ the normalized Hausdorff measure on K, and let $(\mathcal{E}, \mathcal{F})$ be the associated Dirichlet form. Then the eigenvalue counting function $\rho(t)$ of the corresponding Laplacian has the estimate*

$$\rho(t) \asymp t^{\frac{\log 21}{\log 168}}.$$

Proof. We can check the conductances in Theorem 5.1 that (A1), (A2) and (A3) are satisfied, and that $N = 21$ and $\kappa_0 = 8$. Hence Theorem 4.6 yields the conclusion. □

Example 2. Homogeneous forms on 3-dimensional SG

In [11], the authors studied the compatible resistance forms on the 2-dim SG. Let (a_0, b_0, c_0) be the conductance on V_0, and let (a_n, b_n, c_n) be the conductance on V_n. It is shown that in order to have a compatible resistance form, it is necessary and sufficient that the initial conductance satisfies $a_0 \geq b_0 = c_0$ (or their alternations). In the case that $a_0 = b_0 = c_0$, then it gives the canonical form that satisfies the energy self-similar identity; if $a_0 > b_0 = c_0$, then the compatibility of resistances yields

$$a_n \asymp 2^n, \qquad b_n = c_n \asymp \left(\frac{3}{2}\right)^n.$$

It is a degenerated homogeneous RF satisfying (A1)-(A3).

In the following, we consider the 3-dim SG. The situation is far more complicated as there are six choices of conductances on G_0 of V_0. We can not characterize all the homogeneous forms; but we are able to characterize a class with "two-parameter conductances", different from the "one-parameter class" studied previously.

Let p_1, p_2, p_3, p_4 be the four vertices of a unit tetrahedron in $\mathbb{R}^3$. We define the 3-dim SG K to be the unique non-empty compact set in $\mathbb{R}^3$ with the contractions $\{F_i\}_{i=1}^4$ on $\mathbb{R}^3$ such that $F_i(x) = \frac{1}{2}(x-p_i)+p_i$, $1 \leq i \leq 4$. Then K has boundary points $V_0 = \{p_1, p_2, p_3, p_4\}$ and six edges.

We consider the initial conductance (a_0, b_0, c_0), with a_0, c_0 on two opposite edges, and b_0 on the other four edges (see Figure 5). Then let (a_n, b_n, c_n) be the conductance on the n-level cells an in (A1). We will estimate (a_n, b_n, c_n) by the compatibility condition. Let G_n be the corresponding resistance network on V_n.

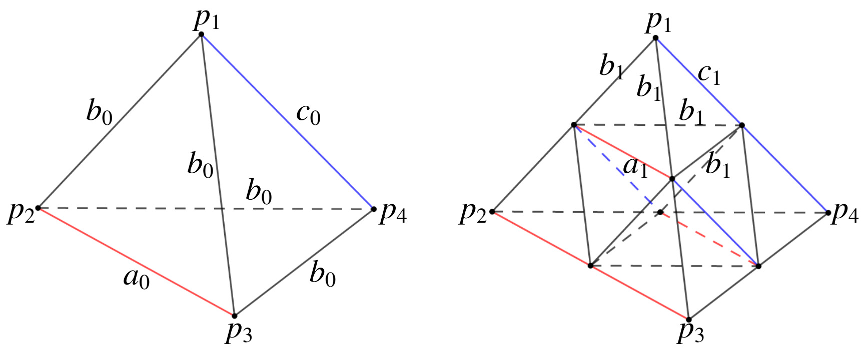

FIGURE 5. The conductances on G_0 and G_1.

To simplify notations, we write (a_0, b_0, c_0) as (A, B, C) and (a_1, b_1, c_1) as (a, b, c) on V_1. According to [21], the associated Laplacian matrix is

$$H_0 = \begin{pmatrix} -2B-C & B & B & C \\ B & -A-2B & A & B \\ B & A & -A-2B & B \\ C & B & B & -2B-C \end{pmatrix},$$

and the trace of G_1 on G_0 is given by $H_1|_{V_0} = T - J^t X^{-1} J$, where

$$T = \begin{pmatrix} -2b-c & & & \\ & -a-2b & & \\ & & -a-2b & \\ & & & -2b-c \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & a & a & 0 \\ b & 0 & b & 0 \\ b & b & 0 & 0 \\ c & 0 & 0 & c \\ 0 & b & 0 & b \\ 0 & 0 & b & b \end{pmatrix},$$

and

$$X = \begin{pmatrix} -2a-4b & b & b & 0 & b & b \\ b & -a-4b-c & a & b & 0 & c \\ b & a & -a-4b-c & b & c & 0 \\ 0 & b & b & -4b-2c & b & b \\ b & 0 & c & b & -a-4b-c & a \\ b & c & 0 & b & a & -a-4b-c \end{pmatrix}.$$

By solving the compatibility condition $H_1|_{V_0} = H_0$ (use Mathematica), we have

$$A = \frac{(a+b)(b^2c + 5abc + 2a^2c + 4b^3 + 9ab^2 + 3a^2b)}{2(a+2b)(4b^2 + 2ac + 3ab + 3bc)}, \tag{5.6}$$

$$B = \frac{2b(a+b)(b+c)}{3ab+4b^2+2ac+3bc}, \tag{5.7}$$

$$C = \frac{(b+c)(ab^2+5abc+2ac^2+4b^3+9b^2c+3bc^2)}{2(2b+c)(4b^2+2ac+3ab+3bc)}. \tag{5.8}$$

By letting $v = a/b, w = c/b$ in the expressions (5.6), (5.8), we define $\Phi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ (here $\mathbb{R}_+ := (0, \infty)$), $\Phi(v, w) = (\Phi_1(v, w), \Phi_2(v, w))$ such that

$$\Phi_1(v, w) := \frac{w(2v^2+5v+1)+3v^2+9v+4}{4(1+w)(2+v)}, \tag{5.9}$$

$$\Phi_2(v, w) := \frac{v(2w^2+5w+1)+3w^2+9w+4}{4(1+v)(2+w)}. \tag{5.10}$$

Note that $\Phi_1(v, w) = \Phi_2(w, v)$ for any $v, w \in \mathbb{R}^2_+$.

For $n \geq 1$, by viewing $(a_{n-1}, b_{n-1}, c_{n-1})$ as (A, B, C), (a_n, b_n, c_n) as (a, b, c), and setting $v_n = \frac{a_n}{b_n}$ and $w_n = \frac{c_n}{b_n}$. The compatibility of the sequence (a_n, b_n, c_n) (to be constructed) implies

$$(v_{n-1}, w_{n-1}) = \Phi(v_n, w_n). \tag{5.11}$$

Remark 1. We will prove that Φ is a one-to-one map on $\mathbb{R}^2_+$ (Lemma 5.6(iii)), and is a homeomorphism on certain sub-region S_∞ (Lemma 5.7(i)). Normally, one will write (5.11) as $\Phi^{-1}(v_{n-1}, w_{n-1}) = (v_n, w_n)$, $\Phi^{-n}(v_0, w_0) = (v_n, w_n)$. However Φ^{-1} has no explicit expression, and is too hard to handle, we will stick to consider $(v_0, w_0) = \Phi^n(v_n, w_n)$ for a fixed initial data.

In the following, we first state a lemma and the main theorem; they will be proved after we give a detailed study of the map Φ in the sequel. Let $S_0 = \{(v, w) : 1 < w < v\}$ and $S_1 = \Phi(S_0)$. Then $S_1 \supset S_0$ (see proof of Lemma 5.7(i)). Inductively, for $n \geq 1$, let $S_n = \Phi(S_{n-1})(\supset S_{n-1})$, and let $S_\infty = \bigcup_{n\geq 0} S_n$; it is invariant under Φ (see Figure 6 for the geometry of this as well as the following lemma).

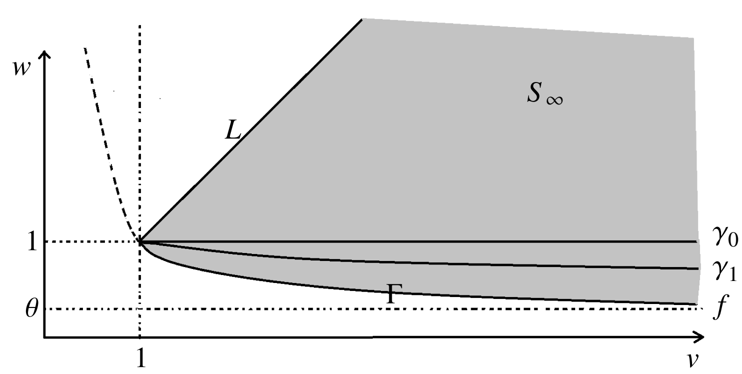

Figure 6. the map Φ on S_∞, the shaded region.

Lemma 5.3. *There exists a continuous non-increasing function $f : [1, \infty) \to (\theta, 1]$ with $\theta = \frac{\sqrt{17}-3}{4}$, and whose graph, denoted by Γ, is on the lower boundary of S_∞ and satisfies $\Phi(\Gamma) = \Gamma$.*

The following is the main theorem of this section. It classifies the initial conductances (a_0, b_0, c_0) that yield compatible resistance forms, and the growth rate of the corresponding conductances $\{(a_n, b_n, c_n)\}$ are also specified.

Theorem 5.4. *Suppose $a_0 \geq c_0 > 0$ and $b_0 > 0$. Let f be the function in Lemma 5.3. Then*

(i) if $\frac{c_0}{b_0} > f\left(\frac{a_0}{b_0}\right)$, then there exists a unique sequence $\{a_n, b_n, c_n\}_{n\geq 0}$ satisfying $a_n \geq c_n$, $\frac{c_n}{b_n} > f\left(\frac{a_n}{b_n}\right)$ and $a_n \asymp c_n \asymp 2^n$, $b_n \asymp 1$;

(ii) if $\frac{c_0}{b_0} = f\left(\frac{a_0}{b_0}\right)$, then there exists a unique positive solution $\{a_n, b_n, c_n\}_{n\geq 0}$ satisfying $a_n \geq c_n$, $\frac{c_n}{b_n} = f(\frac{a_n}{b_n})$; moreover, if $a_0 = c_0$, then $a_n = b_n = c_n = \left(\frac{3}{2}\right)^n a_0$; otherwise if $a_0 > c_0$, then $a_n \asymp 2^n$, $b_n \asymp c_n \asymp \left(\frac{7+\sqrt{17}}{8}\right)^n$;

(iii) if $\frac{c_0}{b_0} < f\left(\frac{a_0}{b_0}\right)$, then no compatible sequence can be constructed.

It follows that for the cases in (ii), the $\{(a_n, b_n, c_n)\}$ defines resistance forms satisfying (A2).

Remark 2. As $v = \frac{a_0}{b_0}$, $w = \frac{c_0}{b_0}$, we see in Figure 6 that (i) is on the grey open region S_∞; (ii) is on the boundary curve Γ, with the first case at the point $(v, w) = (1, 1)$, and the second case are the rest of the points on Γ; (iii) is the region under $\overline{S}_\infty$.

Remark 3. For case (i), since $b_n \asymp 1$, it does not satisfy (A2), hence the induced compatible resistance form is not a homogeneous form. The case in (ii) with $a_0 = c_0$, the conductances $(3/2)^n$ corresponding to the renormalization factor of the standard energy form of 3-dim SG. (Recall that for the d-dim SG, the renormalization factor is $(d+1)/(d+3)$ [19].) For the case $a_0 > c_0$ in (ii), the conductance sequence is analogous to the one in the 2-dim SG, and hence it gives a one-parameter family of limiting diffusion [17].

Remark 4. Note that a_0 and c_0 are conductance on symmetric positions of two edges on the SG. The above theorem holds the same for $c_0 \geq a_0 > 0$ by interchanging the role of a_0 and c_0.

Corollary 5.5. *Suppose $a_0 \geq c_0$ and $\frac{c_0}{b_0} = f\left(\frac{a_0}{b_0}\right)$ as in Theorem 5.4(ii). Then the compatible sequence $\{(a_n, b_n, c_n)\}_{n\geq 0}$ defines a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ where μ is the standard normalized Hausdorff measure on K. Let Δ be the Laplacian of the Dirichlet form and $\rho(t)$ be its eigenvalue counting function. Then for t large enough, we have*

(i). if $a_0 = c_0$, then $\rho(t) \asymp t^{\frac{\log 4}{\log 6}}$;

(ii). if $a_0 > c_0$, then $\rho(t) \asymp t^{\log 4 / \log\left(\frac{7+\sqrt{17}}{2}\right)}$.

Proof. (i) By Theorem 5.4(ii), if $a_0 = c_0$, then $a_n = b_n = c_n = \left(\frac{3}{2}\right)^n \cdot a_0$; a direct check shows that $\kappa_0 = \frac{3}{2}$, and the relation $\sim$ is the trivial relation (each point in V_0 is a single equivalence class). We see that condition (A3) holds. By Theorem 4.6, we obtain, with $N = 4$ and $\kappa_0 = \frac{3}{2}$, that

$$\rho(t) \asymp t^{\frac{\log 4}{\log 6}}.$$

(ii) If $a_0 > c_0$, then by the second conclusion of Theorem 5.4(ii), $a_n \asymp 2^n$, $b_n \asymp c_n \asymp \left(\frac{7+\sqrt{17}}{8}\right)^n$, and thus $\kappa_0 = \frac{7+\sqrt{17}}{8}$, and the equivalence class of $\sim$ is given by $\{p_1\} \cup \{p_2, p_3\} \cup$

$\{p_4\}(=V_0)$. We can check that condition (A3) holds. By Theorem 4.6, we obtain, with $N=4$ and $\kappa_0=\frac{7+\sqrt{17}}{8}$, that

$$\rho(t)\asymp t^{\log 4/\log\frac{7+\sqrt{17}}{2}}.$$

□

To prove Lemma 5.3 and Theorem 5.4, we start by studying some basic properties of the map Φ.

Lemma 5.6. *The map Φ satisfies:*

(i) for $(v,w)\in\mathbb{R}^2_+$, if $\frac{v}{w}=1$, then $\frac{\Phi_1(v,v)}{\Phi_2(v,v)}=1$; and if $\frac{v}{w}>1$, then $1<\frac{\Phi_1(v,w)}{\Phi_2(v,w)}<\frac{v}{w}$;

(ii) for $(v,w)\in\mathbb{R}^2_+$, the partial derivatives satisfy $\frac{\partial\Phi_1}{\partial v}>0$, $\frac{\partial\Phi_1}{\partial w}<0$, $\frac{\partial\Phi_2}{\partial v}<0$, $\frac{\partial\Phi_2}{\partial w}>0$, $\frac{\partial(\Phi_1\cdot\Phi_2)}{\partial v}>0$, $\frac{\partial(\Phi_1\cdot\Phi_2)}{\partial w}>0$;

(iii) Φ is one-to-one from $\mathbb{R}^2_+$ to its image.

Proof. The first statement of (i) follows trivially from (5.9), (5.10). The second statement of (i) follows from a direct calculation:

$$\frac{\Phi_1(v,w)}{\Phi_2(v,w)}=1+\frac{(v-w)(5+2v+2w+vw)(4+3v+3w+2vw)}{(2+v)(1+w)(4+v+9w+5vw+3w^2+2vw^2)}>1$$

and

$$\frac{v}{w}-\frac{\Phi_1(v,w)}{\Phi_2(v,w)}=\frac{(v-w)(2+v+w)(4+v+w)}{w(1+w)(2+v)(4+v+9w+5vw+3w^2+2vw^2)}>0. \tag{5.12}$$

For (ii), let $(v,w)\in\mathbb{R}^2_+$. Then a direct check yields

$$\begin{cases}\frac{\partial\Phi_1}{\partial v}(v,w) &= \frac{14+9w+v(4+v)(3+2w)}{4(2+v)^2(1+w)}>0,\\ \frac{\partial\Phi_1}{\partial w}(v,w) &= -\frac{3+v}{4(1+v)(2+w)}<0.\end{cases} \tag{5.13}$$

As $\Phi_2(v,w)=\Phi_1(w,v)=\Phi(\varphi(v,w))$ where $\varphi(v,w)=(w,v)$, then by the chain rule, $\frac{\partial\Phi_2}{\partial v}(v,w)=\frac{\partial\Phi_1}{\partial w}(w,v)<0$; similarly, $\frac{\partial\Phi_2}{\partial w}(v,w)=\frac{\partial\Phi_1}{\partial v}(w,v)>0$ also.

That $\frac{\partial(\Phi_1\Phi_2)}{\partial v}(v,w)>0$ follows by a direct calculation (by Mathematica) that all the terms are positive (the lengthy expression is omitted). It follows that $\frac{\partial(\Phi_1\Phi_2)}{\partial w}(v,w)=\frac{\partial(\Phi_1\Phi_2)}{\partial v}(w,v)>0$.

Now let us use (ii) to show (iii). Suppose there are two points (v,w) and $(\widetilde{v},\widetilde{w})$ in $\mathbb{R}^2_+$ such that $\Phi(v,w)=\Phi(\widetilde{v},\widetilde{w})$. Without loss of generality, assume $v\le\widetilde{v}$. Let $\eta:[0,1]\to R^2_+$ be the straight line connecting such two points, i.e. $\eta(s)=(s\widetilde{v}+(1-s)v,s\widetilde{w}+(1-s)w)$, $s\in[0,1]$. Then by applying Rolle's theorem to $\Phi_1(\eta(s))$ and $\Phi_1(\eta(s))\Phi_2(\eta(s))$ separately, there exist $s_1,s_2\in(0,1)$ such that

$$\frac{d\Phi_1(\eta(s))}{ds}\Big|_{s=s_1}=0,\qquad \frac{d(\Phi_1(\eta(s))\Phi_2(\eta(s)))}{ds}\Big|_{s=s_2}=0.$$

To prove $(v,w)=(\widetilde{v},\widetilde{w})$, we have two possible cases: *Case (i)* $v=\widetilde{v},w\neq\widetilde{w}$ or $v<\widetilde{v},w\ge\widetilde{w}$: by applying the chain rule to $\frac{d\Phi_1(\eta(s))}{ds}\big|_{s=s_1}$, and using (ii), we have

$$\frac{d\Phi_1(\eta(s))}{ds}\Big|_{s=s_1}=\frac{\partial\Phi_1}{\partial v}(\eta(s_1))\,(\widetilde{v}-v)+\frac{\partial\Phi_1}{\partial w}(\eta(s_1))\,(\widetilde{w}-w)\neq 0,$$

a contradiction. *Case (ii)* $v < \widetilde{v}$ and $w < \widetilde{w}$: by applying the chain rule, and using (ii), we have

$$\frac{d(\Phi_1(\eta(s))\Phi_2(\eta(s)))}{ds}\Big|_{s=s_2} = \frac{\partial(\Phi_1\Phi_2)}{\partial v}(\eta(s_2))\,(\widetilde{v}-v) + \frac{\partial(\Phi_1\Phi_2)}{\partial w}(\eta(s_2))\,(\widetilde{w}-w) > 0,$$

again a contradiction.

In view of the two cases, we conclude that $(v, w) = (\widetilde{v}, \widetilde{w})$, and completes the proof of (iii). □

Next we use the above properties to prove another lemma which illustrates the dynamical behavior of Φ, and plays a key role in obtaining the existence of a compatible sequence $\{(a_n, b_n, c_n)\}_{n\geq 0}$.

Let $S_\infty = \cup_{n=0}^{\infty} S_n$ be the region defined preceding Lemma 5.3. Let $L = \{(v, v) : \ v \geq 1\}$ and $\gamma_0(t) = (t, 1)$, $t \in (1, \infty)$ be the two half-lines. Then the boundary of S_0 is $\partial S_0 = L \cup \gamma_0$.

Lemma 5.7. *The map Φ has the following properties:*
(i) Φ is a homeomorphism between S_∞ and S_∞;
(ii) for $(v, w) \in S_\infty$, we have $v > w$ and $w > \theta$, where $\theta = \frac{\sqrt{17}-3}{4}$ (< 1).

Proof. To prove (i), we note that Φ is a one-to-one continuous map (Lemma 5.6(iii)). By the well-known *Brouwer invariance of domain theorem* [3], we see that S_0 and $S_1 := \Phi(S_0)$ are homeomorphic. We claim that $S_0 \subset S_1$. Indeed, consider $\partial S_0 = L \cup \gamma_0$. First, Lemma 5.6(i) implies $\Phi(L) = L$; secondly, let $\gamma_1 = \Phi(\gamma_0)$, then by the expression of Φ, we have

$$\gamma_1(t) := (\alpha_1(t),\ \beta_1(t)) = \left(\frac{5 + 14t + 5t^2}{8(2+t)},\ \frac{4+2t}{3+3t}\right), \quad t \in (1, \infty). \tag{5.14}$$

It is clear that the range of α_1 is $(1, \infty)$, and $\beta_1(t) < 1$ for all $t > 1$. We conclude that γ_1 is a curve located strictly lower than γ_0. This proves the claim $S_0 \subset S_1$. Inductively, we have $S_{n-1} \subset S_n$, so that $\Phi(S_\infty) = S_\infty$, and Φ is a homeomorphism between S_∞ and S_∞.

To prove (ii), we observe that the statement is trivial on S_0. Suppose the statement is true on S_{n-1}. For $(v, w) \in S_n$, we let $(v', w') \in S_{n-1}$ such that $(v, w) = \Phi(v', w')$. Induction hypothesis implies $v' > w'$. Then the second part of Lemma 5.6(i) yields $v > w$.

To prove $w > \theta$, we note that

$$w = \Phi_2(v', w') = \frac{v'(2w'^2 + 5w' + 1) + 3w'^2 + 9w' + 4}{4(1+v')(2+w')} > \frac{(2w'^2 + 5w' + 1)}{4(2+w')} =: h(w'). \tag{5.15}$$

Consider $h(x) = x$, by solving the equation, we obtain the fixed point $x = \theta = \frac{\sqrt{17}-3}{4}$, which is the positive root of $2x^2 + 3x - 1 = 0$. It is direct to check that $h'(x) > 0$, and hence h is increasing. As $w' > \theta$ (induction hypothesis), we have by (5.11),

$$w > h(w') > h(\theta) = \theta.$$

Hence $w > \theta$ holds for all $(v, w) \in S_\infty = \cup_{n=0}^{\infty} S_n$. □

Proof of Lemma 5.3. We examine the boundary of S_∞ through the boundary of each S_n, and define a continuous function f on $(1, \infty)$ whose graph is the lower part of the boundary of S_∞ (see Figure 6).

For $n \geq 1$, let $\gamma_n(t) = \Phi(\gamma_{n-1}(t))$ and write $\gamma_n(t) = (\alpha_n(t), \beta_n(t))$, $t \in (1, \infty)$. Then use induction, and similar to (5.14), we see that $\alpha_n(t) \to \infty$ as $t \to \infty$. Hence $\alpha_n(t)$ is a homeomorphism from $(1, \infty)$ to itself (by Lemma 5.7(i)).

We claim that *for any $n \geq 1$ and for all $t \in (1, \infty)$, $\alpha_n'(t) > 0$, $\beta_n'(t) < 0$.* We prove this by induction. First, using (5.14) and a direct calculation, we have

$$\alpha_1'(t) = \frac{5}{8} + \frac{3}{8(2+t)^2} > 0, \qquad \beta_1'(t) = -\frac{2}{3(1+t)^2} < 0.$$

Inductively assume that for $k \geq 1$, $\alpha_k'(t) > 0$ and $\beta_k'(t) < 0$. Then by the chain rule and using Lemma 5.6(ii), we have

$$\alpha_{k+1}' = (\Phi_1(\alpha_k, \beta_k))' = \frac{\partial \Phi_1}{\partial v} \cdot \alpha_k' + \frac{\partial \Phi_1}{\partial w} \cdot \beta_k' > 0,$$

and

$$\beta_{k+1}' = (\Phi_2(\alpha_k, \beta_k))' = \frac{\partial \Phi_2}{\partial v} \cdot \alpha_k' + \frac{\partial \Phi_2}{\partial w} \cdot \beta_k' < 0.$$

This completes the proof of the claim.

For $n \geq 1$, since $\alpha_n(t)$ is a homeomorphism from $(1, \infty)$ to $(1, \infty)$, we can define $f_n(v) = \beta_n(\alpha_n^{-1}(v))$, $v \in (1, \infty)$. We claim that

$$|f_n'(v)| \leq C, \ v \in (1, \infty) \ \textit{for some } C > 0, \textit{ independent of } n \textit{ and } v.$$

Indeed, observe that

$$\left|f_{n+1}'\right| = \frac{-\beta_{n+1}'}{\alpha_{n+1}'} = \frac{\left|\frac{\partial \Phi_2}{\partial v}\right| \alpha_n' + \frac{\partial \Phi_2}{\partial w} \left|\beta_n'\right|}{\frac{\partial \Phi_1}{\partial v} \alpha_n' + \left|\frac{\partial \Phi_1}{\partial w}\right| \left|\beta_n'\right|} \leq \frac{\left|\frac{\partial \Phi_2}{\partial v}\right|}{\frac{\partial \Phi_1}{\partial v}} + \frac{\frac{\partial \Phi_2}{\partial w}}{\left|\frac{\partial \Phi_1}{\partial w}\right|}. \tag{5.16}$$

Since $(\alpha_n(t), \beta_n(t))$ is located in the area $(1, \infty) \times (\theta, 1]$, using (5.13), we see that there exists $C_1 > 0$ such that

$$C_1^{-1} \leq \frac{\partial \Phi_1}{\partial v}, \ \left|\frac{\partial \Phi_1}{\partial w}\right| \leq C_1, \quad \text{and} \quad \left|\frac{\partial \Phi_2}{\partial v}\right|, \ \frac{\partial \Phi_2}{\partial w} \leq C_1.$$

This proves the claim.

For fixed $v > 1$, $f_n(v)$ is decreasing on n (as $S_n \subsetneqq S_{n+1}$). Let

$$f(v) = \lim_{n \to \infty} f_n(v). \tag{5.17}$$

By Lemma 5.7(ii), $f_n(v) > \theta$ and clearly $f_n(v) \leq 1$, we see that $\{f_n\}_{n \geq 1}$ are uniformly bounded. By the last claim, $\{f_n\}_{n \geq 1}$ are equi-continuous. Hence by the Arzelà-Ascoli theorem, $f(v)$ is continuous on $(1, \infty)$. Since each f_n is decreasing on v (as $f_n'(v) = \frac{\beta_n'(v)}{\alpha_n'(v)} < 0$), f is non-increasing on v. In addition, $\lim_{v \to 1+} f(v) = 1$.

Finally, let $\Gamma := \{(v, f(v)), v \in (1, \infty)\}$ be the graph of f. Denote by $\overline{S}_\infty$ the closure of S_∞. Then $\partial S_\infty = L \cup \Gamma$ where $L = \{(v, v) : v \geq 1\}$. By Lemma 5.7(i), Φ is a homeomorphism on S_∞, and hence $\Phi(\partial \overline{S}_\infty) = \partial \overline{S}_\infty$. Observe that $\Phi(L) = L$ and $L \cap \Gamma = \emptyset$, and thus we must have $\Phi(\Gamma) = \Gamma$. □

Proof of Theorem 5.4. We follow the preceding notations to write $v_n = \frac{a_n}{b_n}$, $w_n = \frac{c_n}{b_n}$.

Part (i). The assumption implies that $(v_0, w_0) \in S_\infty$, and the sequence $\{(v_n, w_n)\}_{n \geq 0}$ are also in S_∞ (by Lemma 5.7(i)). To prove our result, we need

$$\lim_{n \to \infty} w_n = \infty. \tag{5.18}$$

Granting this, we see from the proof of Lemma 5.6(i) that $v_n > w_n$, and hence $v_n \to \infty$ also; by (5.10) and (5.11), we have

$$w_{n-1} = \frac{v_n(2w_n^2 + 5w_n + 1) + 3w_n^2 + 9w_n + 4}{4(1 + v_n)(2 + w_n)}.$$

A direct estimation shows that for large n,

$$\frac{w_{n-1}}{w_n} = \frac{1}{2}\left(1 + \frac{\varphi(v_n, w_n)}{w_n}\right) < \frac{3}{4},$$

where $\varphi(v_n, w_n)$ includes all the terms of the above equation except the first term; it is a bounded function on $v_n, w_n, n \geq 0$, we use c to denote its bound. We can find $1 < \tau < \frac{4}{3}$ such that $w_n > \tau^n$ for all n. Hence for $n \geq 0$,

$$(1 - c\tau^{-n})\,\frac{w_n}{2} \leq w_{n-1} \leq (1 + c\tau^{-n})\,\frac{w_n}{2}. \tag{5.19}$$

This gives $w_n \asymp 2^n$. By using (5.9) and a similar argument as the above, we also have $v_n \asymp 2^n$.

Now we return to prove (5.18). To do so, we first show that there is $C > 0$ such that for all $n \geq 1$,

$$\frac{v_n}{w_n} \leq C. \tag{5.20}$$

Suppose otherwise. By Lemma 5.6(i), $\frac{v_n}{w_n}$ is increasing in n, and thus we must have $\lim_{n\to\infty} \frac{v_n}{w_n} = \infty$, so that $\lim_{n\to\infty} v_n = \infty$. Then use (5.10) and by the same argument to obtain (5.19). We can conclude that $w_n \asymp 2^n$. The same way, we have $v_n \asymp 2^n$. This contradicts that $\lim_{n\to\infty} \frac{v_n}{w_n} = \infty$. Hence (5.20) holds.

That (5.18) follows by (5.20) can be shown by contradiction. If (5.18) does not hold, then $\{w_n\}_n$ is bounded, so is $\{v_n\}_n$ by (5.20). Let M be their bound. Consider (5.12) for the $\{v_n, w_n\}$. By factoring out the w_n from $(v_n - w_n)$, we can estimate the rest of the terms to be bounded below by a $C > 0$ dependent only on M. Hence

$$\frac{v_n}{w_n} - \frac{v_{n-1}}{w_{n-1}} \geq C\Big(\frac{v_n}{w_n} - 1\Big) \geq C\Big(\frac{v_0}{w_0} - 1\Big). \tag{5.21}$$

By summing up the telescoping sequence on the left, we see that $\frac{v_n}{w_n} \to \infty$ as $n \to \infty$, a contradiction to (5.20). Hence (5.18) holds.

To complete the proof of part (i), we convert the v_n, w_n back to a_n, b_n, c_n. By (5.7), we have

$$\frac{b_{n-1}}{b_n} = \frac{2(v_n + 1)(w_n + 1)}{3v_n + 4 + 2v_n w_n + 3w_n} = 1 + o(\delta^n),$$

for some $\delta \in (0, 1)$, which implies that $b_n \asymp 1$, and hence $c_n = b_n w_n \asymp 2^n$ and $a_n = b_n v_n \asymp 2^n$ as asserted.

Part (ii): As $\Phi(\Gamma) = \Gamma$ (Lemma 5.3), we see that $\{(v_n, w_n)\}_n$ are contained in Γ. For $v_0 > 1$, we first show that $\lim\limits_{n\to\infty} w_n = \theta$. By Lemma 5.6(i), $\frac{v_n}{w_n}\left(= \frac{v_n}{f(v_n)}\right)$ is increasing on n; as f is non-increasing, we see that $(1 >)\{w_n\}_n$ is non-increasing on n. By Lemma 5.7(ii), $w_n \geq \theta$, and thus $\lim_{n\to\infty} w_n = \theta' \geq \theta$ exists. Then by the argument in deriving (5.21), we see that $\frac{v_n}{w_n} \to \infty$ as $n \to \infty$. Hence $v_n \to \infty$. If $\theta' > \theta$, we make use of (5.10) that for large v_n,

$$\frac{w_{n-1}}{w_n} \to \frac{2w_n^2 + 5w_n + 1}{4w_n(2 + w_n)}.$$

Recall that θ is the fixed point of $h(x) = \frac{2x^2+5x+1}{4(2+x)}$ (Lemma 5.7(ii)), a direct calculation shows that $\frac{h(\theta')}{\theta'} < \frac{h(\theta)}{\theta} = 1$. Hence from the above, we have for large v_n, there is $0 < \delta < 1$ such that

$$\frac{w_{n-1}}{w_n} \to \frac{h(\theta')}{\theta'} < \delta < 1.$$

Therefore $w_n \geq C\delta^{-n} \to \infty$, a contradiction, and we conclude that $\lim_{n\to\infty} w_n = \theta$.

By (5.9), for n large,

$$\frac{v_n}{v_{n-1}} \to \frac{4+4w_n}{3+2w_n} > 1,$$

so that there is some $\tau > 1$ and $C > 0$ such that for n large enough,

$$v_n \geq C^{-1}\tau^n. \tag{5.22}$$

Substituting this into

$$\frac{w_{n-1}}{w_n} = \frac{v_n(2w_n^2 + 5w_n + 1) + 3w_n^2 + 9w_n + 4}{4w_n(1+v_n)(2+w_n)},$$

we have

$$\frac{2w_n^2 + 5w_n + 1}{4w_n(2+w_n)} \geq 1 - C'\tau^{-n},$$

(It seems we need this to get that w_n goes to θ sufficiently fast)

which implies that

$$w_n - \theta \leq C''\tau^{-n}.$$

Therefore, from this and (5.22), we have for n large enough,

$$\frac{v_n}{v_{n-1}} = \frac{4+4w_n}{3+2w_n} + O(\tau^{-n}) \to \frac{4+4\theta}{3+2\theta},$$

which implies that $v_n \asymp \left(\frac{4+4\theta}{3+2\theta}\right)^n = \left(\frac{7-\sqrt{17}}{2}\right)^n$.

In the case $a_0 = c_0$, then we must have $b_0 = a_0 = c_0$ also (for if otherwise, $b_0 = \eta^{-1}a_0$, then $v_0 = w_0 = \eta$, but (η,η) is not on Γ); hence $v_n = w_n = 1, n \geq 0$. From (5.7), we have $\frac{b_{n-1}}{b_n} = \frac{2}{3}$. Hence $a_n = b_n = c_n = \left(\frac{3}{2}\right)^n a_0$.

Assume that $a_0 > c_0$. By (5.7) and the above estimate, we have

$$\frac{b_{n-1}}{b_n} = \frac{2(v_n+1)(w_n+1)}{3v_n + 4 + 2v_nw_n + 3w_n} = \frac{2+2w_n}{3+2w_n} + o(\delta^n) = \frac{2+2\theta}{3+2\theta} + o(\delta^n),$$

for some $\delta \in (0,1)$, we obtain $b_n \asymp \left(\frac{3+2\theta}{2+2\theta}\right)^n$ where $\theta = \frac{\sqrt{17}-3}{4}$. Hence $c_n = b_nw_n \asymp b_n \asymp \left(\frac{7+\sqrt{17}}{8}\right)^n$, and $a_n = b_nv_n \asymp \left(\frac{3+2\theta}{2+2\theta}\right)^n \left(\frac{4+4\theta}{3+2\theta}\right)^n \asymp 2^n$.

Proof of (iii): Let (v_0, w_0) satisfy $f(v_0) > w_0$, and assume that there is a compatible sequence $\{(v_n, w_n)\}_{n\geq 0}$. As Φ is a homeomorphism on S_∞, we see that $1 > f(v_n) > w_n$. Consider $h(x) = \frac{2x^2+5+1}{4(2+x)}$ on $[0,1]$. Note that $0 < h'(x) < 1$, so that $h(x)$ is an increasing function. By (5.15), we see that $w_0 > h(w_1)$, and repeatedly applying h, we have

$$w_0 > h(w_1) > h^{(2)}(w_2) > \cdots > h^{(n)}(w_n).$$

Also note that

$$|h^{(n)}(w_n) - \theta| = |h^{(n)}(w_n) - h^{(n)}(\theta)| \leq c^n|w_n - \theta|,$$

where $c = \max_{x\in[0,1]}\{h'(x)\} < 1$, which yields

$$w_0 > h^{(n)}(w_n) \to \theta.$$

Note that this is also true if we start from any $(v_m, w_m), m < n$ instead of (v_0, w_0). Hence we can conclude that

$$1 > f(v_n) > w_n \geq \theta. \tag{5.23}$$

We also note that by applying the same argument as in (5.21), then $v_n/w_n \to \infty$ as $n \to \infty$, and hence $v_n \to \infty$ holds (as $w_n < 1$).

Next we define a sequence of positive numbers $d_n := |f(v_n) - w_n| = f(v_n) - w_n$ for $n \geq 0$. We want to show that there is $\delta \in (0, 1)$ such that

$$d_n > \delta^{-1} d_{n-1}. \tag{5.24}$$

This will lead to a contradiction, as both $\{f(v_n)\}_n$ and $\{w_n\}_n$ are bounded. Denote by $p = (v_n, f(v_n))$, $q = (v_n, w_n)$. Then $f(v_n) > w_n \geq \theta$. Write

$$\Phi(p) = (\Phi_1(v_n, f(v_n)), \Phi_2(v_n, f(v_n))), \qquad \Phi(q) = (\Phi_1(v_n, w_n), \Phi_2(v_n, w_n)).$$

These are points satisfying $\Phi(p) \in \Gamma$, $\Phi(q) = (v_{n-1}, w_{n-1})$. (See Figure 7.)

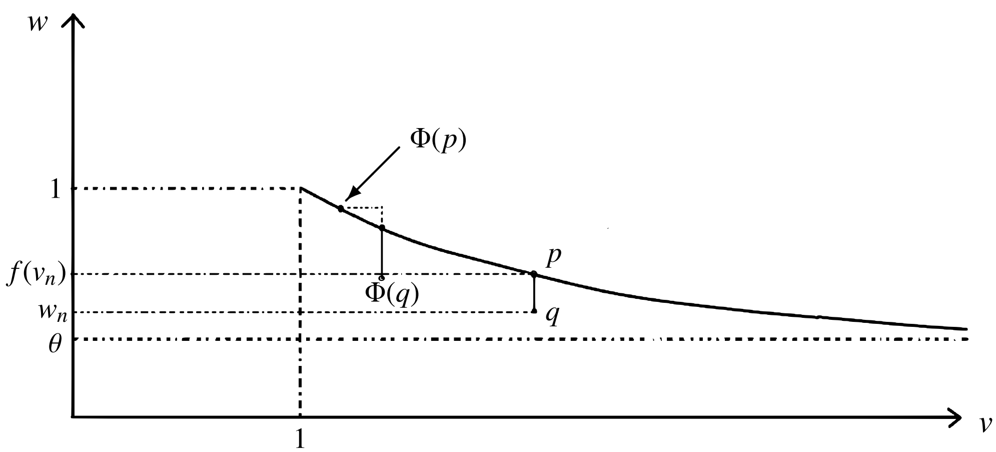

FIGURE 7. The positions of p, q, $\Phi(p)$ and $\Phi(q)$.

Consider $\frac{\partial \Phi_2}{\partial w}$ (use (5.13) and $\Phi_2(v, w) = \Phi_1(w, v)$). It follows from $v \to \infty$ and $w \to \theta$ (from the last statement in the last paragraph) that

$$\frac{\partial \Phi_2}{\partial w} \to \frac{9 + 8\theta + 2\theta^2}{4(2 + \theta)^2} = \frac{5\theta + 10}{10\theta + 18} < 1. \tag{5.25}$$

As $\frac{\partial \Phi_2}{\partial w} > 0$, and by (5.25), for v large and w close to θ, there is $0 < \delta < 1$ such that

$$d_{n-1} = f(v_{n-1}) - w_{n-1} = \int_{w_n}^{f(v_n)} \frac{\partial \Phi_2}{\partial w_n} dw < \delta(f(v_n) - w_n) = \delta d_n.$$

This proves (5.24), and completes the proof of part (iii). □

We remark that in the above proof of (iii), we consider $(v_0, w_0) \notin \overline{S}_\infty$ with $v_0 > w_0$, and look at the reverse iteration $(v_n, w_n) = \Phi^{-n}(v_0, w_0)$. Then from (5.24), we see after finitely many steps $\Phi^{-n}(v_0, w_0)$ will go outside the positive quadrant. Hence for such initial condition, not even (A1) can be established.

In view of Remark 4 after Theorem 5.4, we have settled the existence of DF for all the cases $(v, w) \in \mathbb{R}_+^2$ except at $I = (0, 1)^2$. However this part can also be resolved following our line of investigation. Indeed, if we consider Φ on $\mathbb{R}^2_+$, then $\Phi^n(\mathbb{R}^2_+)$ will shrink to $\mathcal{S}_\infty := S_\infty \cup S^t_\infty \cup L$ where S^t_∞ is the reflection of S_∞ along the diagonal line $v = w$, and $\Phi^n(I)$ will converge to $\{(1,1)\}$. If we pick a $(v_0, w_0) \in I$, then $\Phi^{-n}(v_0, w_0)$ will eventually map outside $\mathbb{R}^2_+$ by a similar argument as in the proof of (iii).

6. A basic assumption in [13]

In [13], the generalized d-dimensional ℓ-level SGs and the heat kernels of the "asymptotically lower-dimensional diffusions" were considered. The model was given conductances on G_0 that is 1 on the horizontal edges, and w on the slant edges, and by the homogeneity condition (A1), the conductances on the n-th iteration G_n is defined (see Figure 8), that is, the horizontal face of a subcell is a $(d-1)$-dim SG and has equal conductance on all its edges, and the conductance of the slant edges and the horizontal edges has ratio w and 1. This is the "one-parameter family" model (on w) in [13], extending the one in [17] on the 2-dim SG, which was proved in [11] to be the only compatible resistance form on SG (under the assumption of homogeneity (A1)).

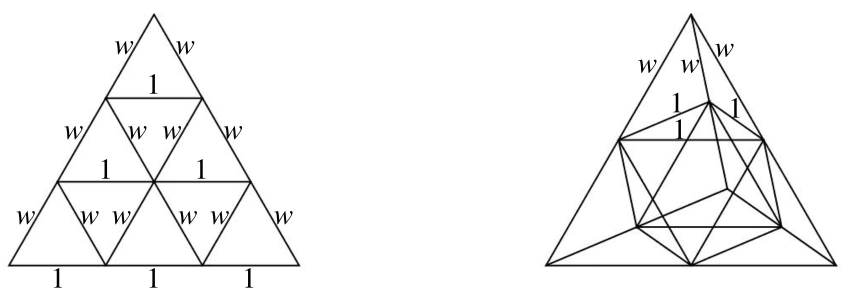

Figure 8. The d-dimensional SG's with ℓ-level, $(d, \ell) = (2, 3)$ and $(3, 2)$.

Denote this network by D_w. It was shown that one can write the trace map $T - J^t X^{-1} J = R(w)^{-1} D_{\alpha(w)}$, where $R(w)$ is the resistance scaling factor of the horizontal edges of the $(d-1)$-dim SG, and $\alpha(w)$ is the new conductance of edges connecting vertices of different levels. The map $\alpha : [0,1] \to [0,1]$ is a strictly increasing map with two fixed points 0 and 1, where the case $w = 1$ is corresponding to the standard resistance form on K. The existence of such form was proved in [23], [21].

To put the conductances of the model into the notations as in §5, Example 2, on each subcell in G_n, we let (a_n, b_n) denote the conductances of the horizontal edges of the $(d-1)$-dim SG, and the slant edges connecting each level respectively. It was estimated in [13, Lemma 3.11] that $a_n \asymp R_G^n$ where $R_G(> 1)$ is the reciprocal of the renormalization factor of the standard form on a $(d-1)$-dim SG. By factoring out a_n, the $\frac{b_n}{a_n}$ is related to the $\alpha(w)$ as follows: let $\alpha^{-1}(w)$ denote the inverse map of $\alpha(w)$. By using the reverse recursive construction, if we start at w, then $\frac{b_n}{a_n} = \alpha^{-n}(w)$. It was proved in [13, Lemma 3.9] that $\alpha^{-n}(w) \asymp \beta^n$ for some $\beta \in (0,1)$. Hence we have

$$a_n \asymp R_G^n, \qquad b_n \asymp (R_G \beta)^n. \tag{6.1}$$

In [13, p.373], the authors made the following assumption.

Assumption-(HK): $R_G\beta > 1$.

In view of (6.1), $a_n > b_n \asymp R_G\beta > 1$, hence Assumption-(HK) implies (A2) and defines a homogeneous resistance form. Hence, consider the limiting process in small scale, it was proved that a random walk will converge asymptotically to the horizontal $(d-1)$-dim hyperplanes in K where their edges in G_n have conductance a_n.

In [13], it was checked that the assumption holds for cases of $d, \ell \leq 3$. In the following, using some resistor network techniques, we can show that Assumption-(HK) always holds. Moreover, we can identify the precise anisotropic growth rate of the conductances $\{b_n\}_n$.

To this end, we fist give some notations. Let K be a d-dimensional SG with ℓ levels, $d \geq 2$ and $\ell \geq 2$. On V_1, for $1 \leq i \leq \ell$, let $\mathcal{N}(d,i)$ denote the number of horizontal $(d-1)$-dim SGs in the i-th level of V_1 (counting from the top downward). This equals the number of 1-cells in V_1 sitting on the i-th level. By convention, we take $\mathcal{N}(1,i) = 1$ for all $i \geq 1$. We see that for $d \geq 2$,

$$\mathcal{N}(d,1) = 1, \quad \text{and} \quad \mathcal{N}(d,i) = \sum\nolimits_{j=1}^{i} \mathcal{N}(d-1,j), \qquad i \geq 1.$$

Let H_i, $0 \leq i \leq \ell$ denote the vertices on the i-th level (H_0 is the top vertex of the d-dim SG), and denote by $x \sim y$ for $x \in H_{i-1}$, $y \in H_i$ if x and y are connected by a slant edge. Then it is seen that the number of such pairs of the two levels is $d\mathcal{N}(d,i)$.

Since we are fixing $d \geq 2$ in the following, we will write $\mathcal{N}_i := \mathcal{N}(d,i)$ for simplicity.

Proposition 6.1. *Let K be a d-dimensional ℓ-level SG, let $0 < w < 1$ and $(a_0, b_0) = (1, w)$ be the conductances on V_0. Then*

$$a_n \asymp R_G^n, \qquad b_n \asymp (R_G\beta)^n \asymp \left(\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}\right)^n.$$

It follows that $R_G\beta > 1$.

Proof. In view of (6.1), we only need to estimate b_n. By the homogeneity condition (A1), for simplicity, we consider network G_0 on V_0 with conductance (a_n, b_n), and G_1 on V_1 with conductance (a_{n+1}, b_{n+1}). By shorting the d points in V_0 contained in the bottom face, and by the compatibility, it is seen that the effective conductance between the top vertex P and the bottom in V_0 is $db_n =: C_n$ by the parallel law.

Recall the monotonicity law says that if we decrease the resistance somewhere in the network, then the resistances of any two nodes will not increase [7]. Hence the conductance C_{n+1} of G_1 becomes larger if we shorten the horizontal edges in each level of V_1 to a node, and the trace on V_0 will have larger conductance than C_n. Denote this effective conductance by C_n'. Then by using the parallel law, we see that the resistance between any two levels H_{i-1}, H_i is $\frac{1}{b_{n+1} \cdot d\mathcal{N}_i}$. The series law now implies that the effective resistance on G_0 is $\sum_{i=1}^{\ell} \frac{1}{b_{n+1} \cdot d\mathcal{N}_i}$. Therefore we obtain

$$C_n' := db_{n+1} \left(\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}\right)^{-1}.$$

Hence by $C'_n \geq C_n$,

$$\frac{b_{n+1}}{b_n} \geq \sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i} \qquad n \geq 0. \tag{6.2}$$

Observe that $b_n = (b_n/b_{n-1}) \cdots (b_2/b_1) \cdot b_1$. Applying the above to $1 \leq k < n$, we have $b_n \geq b_0 (\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i})^n$. As $\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i} > \frac{1}{\mathcal{N}_1} = 1$, and $(R_G\beta)^n \asymp b_n$, we see that $R_G\beta \geq \sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i} > 1$. This confirms Assumption-(HK) always hold for the generalized SG model in [13].

In the following we prove that $R_G\beta$ actually equals $\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}$. For (a_{n+1}, b_{n+1}) on G_1, we consider the harmonic function u on V_1 with values 1 at the top vertex P and 0 at B_0, the d points in V_0 contained in the bottom face. Then the graph energy $E(u) = db_n$ by the compatibility. Consider a pair of horizontal edges x, y in a level H_i in G_1. We then have $a_{n+1}|u(x) - u(y)|^2 \leq E(u)$, and hence $|u(x) - u(y)|^2 \leq db_n/a_{n+1}$. As a consequence, $|u(x) - u(y)|^2$ is of order at most $O(\beta^n)$ for any $x, y \in H_i$.

For each $1 \leq i \leq \ell$, fix an $x_i \in H_i$. By the above estimate, it is direct to check that for $x \in H_{i-1}, y \in H_i$,

$$\begin{aligned} |u(x) - u(y)|^2 &= |u(x_{i-1}) - u(x_i) + (u(x) - u(x_{i-1}) + u(x_i) - u(y))|^2 \\ &\geq |u(x_{i-1}) - u(x_i)|^2 - |u(x_{i-1}) - u(x_i)| \cdot O(\beta^{n/2}) \end{aligned} \tag{6.3}$$

where $|O(\beta^{n/2})| \leq C\beta^{n/2}$ for some $C > 0$ (independent of n).

Write $E(u)$ into the summation of horizontal energies $E_i^{(h)}(u)$ on each layer H_i for $i = 1, \cdots, \ell$, and the slant energies $E_i^{(s)}(u)$ between two neighboring layers H_{i-1} and H_i for $i = 1, \cdots, \ell$. Note that the number of the pairs $x \sim y$ for $x \in H_{i-1}$ and $y \in H_i$ is $d\mathcal{N}_i$. By using (6.3), we have,

$$\begin{aligned} db_n = E(u) &= \sum\nolimits_{i=1}^{\ell} E_i^{(h)}(u) + \sum\nolimits_{i=1}^{\ell} E_i^{(s)}(u) \\ &\geq \sum_{i=1}^{\ell} \sum_{\substack{x \in H_{i-1}, y \in H_i \\ x \sim y}} b_{n+1} \big(u(x) - u(y)\big)^2 \\ &\geq \sum\nolimits_{i=1}^{\ell} d\mathcal{N}_i \, b_{n+1} \Big(u(x_{i-1}) - u(x_i)\Big)^2 - b_{n+1} \cdot O'(\beta^{n/2}) \end{aligned} \tag{6.4}$$

where $|O'(\beta^{n/2})| \leq C'\beta^{n/2}$ for some $C' > 0$ independent of n.

Now we view $x_0 (= P), x_1, \cdots, x_\ell$ as a one-dimensional network with resistance $\frac{1}{\mathcal{N}_i}$ between two nodes, and the effective resistance on x_0 and x_ℓ is $\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}$. By $u(x_0) = 1$, $u(x_\ell) = 1$, and the minimal energy property of harmonic extension, it follows that (treating $u(x_i)$'s as an extension of u),

$$\sum_{i=1}^{\ell} \mathcal{N}_i \, |u(x_{i-1}) - u(x_i)|^2 \geq \Big(\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i} \Big)^{-1} =: Q^{-1}. \tag{6.5}$$

Also letting $\varepsilon_n = C' d^{-1} Q \beta^{n/2}$, and substitute this into (6.4), we obtain

$$b_n \geq b_{n+1} \cdot Q^{-1}(1 - \varepsilon_n), \quad i.e. \quad \frac{b_{n+1}}{b_n} \leq Q(1 - \varepsilon_n)^{-1}.$$

Apply this to $1 \le k < n$, and combine with (6.2), we have

$$Q^n b_0 \le b_n \le Q^n b_0 \prod_{k=0}^{n-1} (1-\varepsilon_k)^{-1}.$$

It is standard to check that $\prod_{k=0}^{n-1}(1-\varepsilon_k)$ converge to a constant: $\prod_{k=0}^{n-1}(1-\varepsilon_k) = e^{\sum_{k=0}^{n-1}\log(1-\varepsilon_k)} \asymp e^{-\sum_{k=0}^{n-1}\varepsilon_k} \to e^{-c/(1-\beta^{1/2})}$ as $n \to \infty$ for some $c > 0$. Hence we have

$$b_n \asymp Q^n = \Big(\sum_{i=1}^{\ell} \frac{1}{\mathcal{N}_i}\Big)^n$$

as asserted. □

It is direct to apply Theorem 4.6 and Proposition 6.1 to obtain the spectral asymptotics for this class of Dirichlet forms:

$$\rho(t) \asymp t^{\frac{\log \mathcal{N}(d+1,\ell)}{\log(\mathcal{N}(d+1,\ell)S(d,\ell))}},$$

where $S(d,\ell) = \sum_{i=1}^{\ell} \frac{1}{\mathcal{N}(d,i)}$.

In [13], the authors considered another class of examples called Vicsek checkerboards (VC), they observed that in some cases, Assumption $R_G\beta > 1$ is not satisfied. Indeed, by modifying the about shorting technique on resistance networks, it is not hard to show that Assumption $R_G\beta \le 1$ in all situations. We omit the details here.

7. Remarks and future work

In (5.1), we have indicated that the basic idea of the reverse recursive construction of anisotropical DF is that if the trace map Φ is invertible, then we can consider

$$\Phi^{-n}(\mathbf{x}_0) = \mathbf{x}_n,$$

where $\mathbf{x}_0 \ge 0$ is the initial resistances (conductances), that its coordinates form a Laplacian matrix [21] on V_0 (i.e., on the edges of G_0). If the sequence $\mathbf{x}_n$ has an asymptotic growth rate > 1, that is Φ^{-1} can iterate continuously, and the $\mathbf{x}_n$ will give a compatible sequence of resistance (or conductance) on G_n, which yields a homogeneous resistance form. In fact this is used in [17, 11, 13] and the eyebolted Vicsek cross (Theorem 5.1) to obtain such a form. However, this vastly oversimplifies the problem; in general, the map Φ may not be one-to-one, and even so, Φ^{-1} may only be dealt with implicitly. Also, the choice of the initial data is important as the compatible sequence may not have an asymptotic growth rate > 1. Hence additional techniques have to be introduced; many of these are encountered in the proof of Theorem 5.4 of the 3-dim SG.

In the 3-dim SG (§5, Example 2), the network resistance network G_n on V_0 is a two-parameter family $\{(v_n, w_n)\}_n$. After eliminating all the cases, there is only one compatible sequence of conductance left, which gives an asymptotically one-dimensional diffusion as in [17] (see Theorem 5.4 and Remarks 2, 3, 4). On the other hand in [13], their example is based on a one-parameter family $\{w_n\}_n$, which gives an asymptotically $(d-1)$-dimensional diffusion. We ask if we can exhaust all the degenerated RF on 3-dim SG or whether the two types are the only "degenerated" homogeneous forms. With an analogous two-parameter family setting up as §5 Example 2, the same question can be asked for the general d-dim, ℓ-level SG studied in [13]. It will also be interesting to study the other well-known nested fractals, such as the pentagasket, snowfake.

To get a better understanding of the reverse recursive construction in the more general situation, it is essential to investigate the questions of Φ^{-1} in the first paragraph. In particular, it is important to find nice sufficient condition to guarantee that the map Φ is one-to-one.

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