A SIERPINSKI CARPET LIKE FRACTAL WITHOUT STANDARD SELF-SIMILAR ENERGY

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Abstract. We construct a Sierpinski carpet like fractal, on which a diffusion with sub-Gaussian heat kernel estimate does not exist, in contrast to previous researches on the existence of such diffusions, on the generalized Sierpinski carpets and recently introduced unconstrained Sierpinski carpets.

1. Introduction

In history, the existence of locally symmetric diffusions with sub-Gaussian heat kernel estimates on Sierpinski carpets (SC) was proved in the pioneering works [1, 2, 3] by Barlow and Bass, using a probabilistic method. One key step in the proof is to show certain paths, named 'Knight moves' and 'corner moves', occur with positive probability, and the argument relies on the local symmetry of the fractals. By introducing the difficult coupling argument, the result was later extended to generalized Sierpinski carpets (GSC) [4], which are higher dimensional analogues of SC.

A different approach, later shown in [5] to yield the same diffusion as that of Barlow and Bass in [1, 4], was introduced by Kusuoka and Zhou [18]. The strategy is to construct self-similar Dirichlet forms on fractals as limits of averaged rescaled energies on cell graphs. This approach is analytic, except a key step to verify that the resistance constants and the Poincare constants are comparable. In particular, the probabilistic 'Knight move' argument was borrowed to achieve this, see Theorem 7.16 in [18]. This gap was fulfilled in the recent work [7] of the authors by a chaining argument of resistances, and the existence theorem extends to a class of carpet-like fractals, named unconstrained Sierpinski carpets (USC). In some sense, the USC are more flexible in geometry as cells except those along the boundary are allowed to live off the grids. See Figure 1 for examples of SC and USC. An analytic approach on GSC when the Hausdorff dimension is greater than 2 still remains unknown.

It is believed by the authors that the same approach may be adapted to some planar fractals with relaxed boundary restriction, or even with contraction ratios of 1-cells allowed to be distinct, but still keeping the high symmetry. This naturally leads us to look at a larger family, named as Sierpinski carpet like fractals (LSC), see Definition 2.1. Unexpectedly, in this paper, we will present an example of LSC, which does not have a 'nice' self-similar Dirichlet form. Precisely, there does not exist a diffusion (even without the self-similarity requirement)

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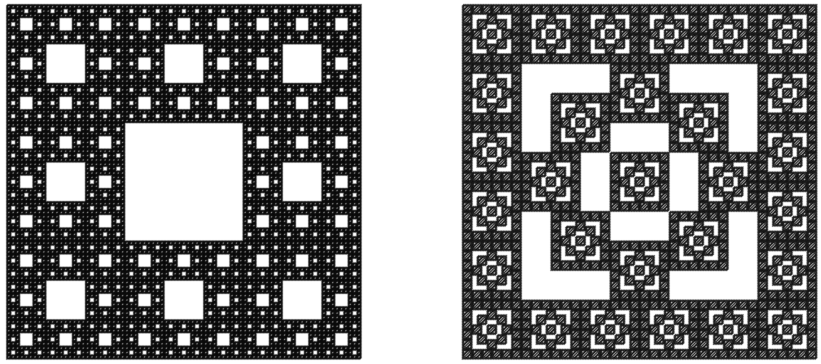

FIGURE 1. Two examples of USC (the left one is the standard SC).

with sub-Gaussian heat kernel estimate on this fractal. In addition, this example can also be easily generalized to the higher dimensional case.

Before proceeding, we briefly explain the analytic approach to the existence [18, 7]. This will help readers to see the intuition that motivate the construction (see Remark 3.6). In [18], for a large class of fractals, four sequences of constants were introduced, and the two most important ones are the *Poincare constants* λ_n and the *resistance constants* R_n (we adapt λ_n here by a scaling to help readers to see the relation):

$$\lambda_n = \sup_{f\in l^2(W_n)} \frac{[(f-[f]_{W_n})^2]_{W_n}}{E_n(f)},$$

where W_n denotes the set of n-cells of the fractal, $[f]_{W_n}$ denotes the average of f on W_n, and E_n is the natural cell graph energy on W_n;

$$R_n = \inf_{w\in W_m, m\geq 1} R_{m+n}(w\cdot W_n, C_w\cdot W_n),$$

where $R_{m+n}(X,Y)$ is the effective resistance associated with the energy form E_{m+n} between two subsets X,Y in W_{m+n}, $w\cdot W_n$ is the set of $(m+n)$-cells contained in w, C_w consists of those m-cells that are not neighboring to w, and $C_w\cdot W_n$ is the set of $(m+n)$-cells contained in C_w. It is natural to see $\lambda_n \gtrsim R_n$, and moreover for the recurrent case in [18] (also see [7] for the reorganized version where the constants λ_n^D and some related estimates are dropped), it is proved that

$$R_n\lambda_m \lesssim \lambda_{n+m} \lesssim \lambda_n\lambda_m.$$

To achieve an exponential growth order of R_n, we need a remaining piece of estimate $\lambda_n \lesssim R_n$. This was not proved analytically until recently in [7]. The proof is based on the fact that $\lambda_n \geq \rho^m\lambda_{n-m}$ for some $\rho > 1$ (for the lower dimensional case), so $\lambda_n \gg \lambda_{n-m'}$ if we fix some m' large enough. In particular, if we choose f such that $E_n(f) = 1$ and $[f^2]_{W_n} - [f]^2_{W_n} = \lambda_n$, then the variation of f on each $(n-m')$-cell (and also neighboring n-cells) is negligible with respect to the total variation of f. Using this convenience, on a SC, one can pick two $(n-m')$-cells with difference of f comparable with $\lambda_{n-m'}$, and thus the resistance between opposite boundary lines can be estimated from below in terms of λ_n. On USC, based on the same idea, a more complicated argument can be applied to find a chain of $(n-2m')$-cells, lying along the boundary of some $(n-m')$-cell. In addition, a linearization extension argument is provided to fulfill the desired resistance estimate. See [7, Section 4] for details.

The contents of this paper are as follows. In Section 2, we present the exact definition of LSC and list some necessary notations. In Section 3, we construct the counter-example of LSC as desired. In Section 4, we prove that on a LSC fractal a conservative regular Dirichlet form with sub-Gaussian heat kernel estimate will automatically induce an equivalent standard self-similar one (see Definition 2.2).

2. Sierpinski carpet like fractals

We will consider carpet-like fractals, which are generated with a similar iterated function system (IFS for short) as SC, living symmetrically in a square, with more flexible locations of the cells (copies of the fractal under the composition of mappings in the IFS) as USC, but allowing the contraction ratios of the IFS to be distinct.

Let $\square := [0,1]^2$ be the unit square in $\mathbb{R}^2$. We let

$$q_1 = (0,0), \quad q_2 = (1,0), \quad q_3 = (1,1), \quad q_4 = (0,1)$$

be the four vertices of $\square$, where we write $x = (x_1, x_2)$ for a point in $\mathbb{R}^2$. In addition, we will write

$$\overline{x,y} = \big\{(1-t)x + ty : 0 \le t \le 1\big\},$$

for the line segment connecting points x, y in $\mathbb{R}^2$.

For convenience, we denote the group of self-isometries on $\square$ by

$$\mathscr{G} = \big\{\Gamma_v, \Gamma_h, \Gamma_{d_1}, \Gamma_{d_2}, id, \Gamma_{r_1}, \Gamma_{r_2}, \Gamma_{r_3}\big\},$$

where $\Gamma_v, \Gamma_h, \Gamma_{d_1}, \Gamma_{d_2}$ are *reflections* (v for vertical, h for horizontal, d_1, d_2 for two diagonals),

$$\begin{aligned} &\Gamma_v(x_1,x_2) = (x_1, 1-x_2), \qquad \Gamma_h(x_1,x_2) = (1-x_1, x_2),\\ &\Gamma_{d_1}(x_1,x_2) = (x_2, x_1), \qquad \Gamma_{d_2}(x_1,x_2) = (1-x_2, 1-x_1), \end{aligned} \tag{2.1}$$

for $x = (x_1,x_2) \in \square$; id is the identity mapping; and $\Gamma_{r_1}, \Gamma_{r_2}, \Gamma_{r_3}$ are *rotations*,

$$\Gamma_{r_1}(x_1,x_2) = (1-x_2, x_1), \quad \Gamma_{r_2} = (\Gamma_{r_1})^2, \quad \Gamma_{r_3} = (\Gamma_{r_1})^3, \tag{2.2}$$

around the center of $\square$ counter-clockwisely with angle $\frac{j\pi}{2}$, $j = 1,2,3$.

We will always require all the structures (eg: the Dirichlet forms, the diffusions, the measures, etc.) under consideration are $\mathscr{G}$-symmetric without explicit mention.

Definition 2.1 (Sierpinski carpet like fractals (LSC))**.**

Let $N \ge 8$ and $\{\rho_i\}_{i=1}^N$ be a collection of positive numbers with $\sum_{i=1}^N \rho_i^2 < 1$. Let $\{F_i\}_{1\le i\le N}$ be a collection of similarities with the form $F_i x = \rho_i x + c_i$ for some $c_i \in \mathbb{R}^2$. Assume that the following hold:

(Non-overlapping). $F_i(\square) \cap F_j(\square)$ is either a line segment, or a point, or empty, $i \ne j$;
(Connectivity). $\bigcup_{i=1}^N F_i(\square)$ is connected;
(Symmetry). $\Gamma\big(\bigcup_{i=1}^N F_i(\square)\big) = \bigcup_{i=1}^N F_i(\square)$ for any $\Gamma \in \mathscr{G}$;
(Boundary included). $\overline{q_1,q_2} \subset \bigcup_{i=1}^N F_i(\square) \subset \square$.

We call the unique compact subset $K \subset \square$ satisfying

$$K = \bigcup_{i=1}^{N} F_i K$$

a *Sierpinski carpet like fractal (LSC)*.

Throughout the paper, we always let d be the Euclidean metric on K induced from $\mathbb{R}^2$. Note that by the non-overlapping condition, the invariant set K satisfies the open set condition, i.e. $\bigcup_{i=1}^{N} F_i(\square^\circ) \subset \square^\circ$ and $F_i(\square^\circ) \cap F_j(\square^\circ) = \emptyset$ for each pair $i \neq j$, where $\square^\circ$ denotes the interior of $\square$. It follows that the Hausdorff dimension of K, denoted as d_H, is the unique solution $\alpha \in \mathbb{R}$ of $\sum_{i=1}^{N} \rho_i^\alpha = 1$. In addition, the d_H-dimensional Hausdorff measure of K is positive and finite. We will always let μ be the normalized d_H-dimensional Hausdorff measure on K. That is, $\mu(F_i K) = \rho_i^{d_H}$ for each i, and $\mu(F_w K) = \rho_w^{d_H}$, where $\rho_w := \rho_{w_1} \cdots \rho_{w_m}$, for each $w = w_1 \cdots w_m \in W_m := \{1, \cdots, N\}^m$ with $m \geq 0$.

The condition $N \geq 8$ is a requirement of the boundary included condition. The condition $\sum_{i=1}^{N} \rho_i^2 < 1$ ensures that $d_H < 2$, so that we are dealing with a non-trivial planar self-similar set.

Note that when $k = 3$, $N = 8$ and all $\rho_i = \frac{1}{3}$, K is the standard SC. Comparing LSC with USC introduced in [7], the main difference is that, the contraction ratios of the IFS are kept to be the same in the latter.

Our naive question is to ask whether there always exist natural 'nice' diffusion processes on the more flexible LSC. It was proved in [3, 4, 7] that the self-similar diffusions on SC, GSC and USC, where $\rho_i = k^{-1}$ for any $1 \leq i \leq N$ for some $k \in \mathbb{N}$ with $k \geq 3$, always enjoy a sub-Gaussian heat kernel estimate,

$$\begin{aligned} \frac{c_1}{t^{d_H/\beta}} \exp(-c_2 (\frac{d(x,y)^\beta}{t})^{\frac{1}{\beta-1}}) &\leq p(t,x,y) \\ &\leq \frac{c_3}{t^{d_H/\beta}} \exp(-c_4 (\frac{d(x,y)^\beta}{t})^{\frac{1}{\beta-1}}), \quad \forall 0 < t \leq 1, \forall x, y \in K, \end{aligned} \tag{2.3}$$

where $p(t,x,y)$ is the *heat kernel* (also called *transition density*), $\beta = -\frac{\log \eta}{\log k} + d_H$ is the *walk dimension*, $0 < \eta < 1$ is the common renormalization factor of the self-similar diffusion, and $c_1, \ldots, c_4$ are positive constants. To be precise, we need to find certain self-similar Dirichlet forms on LSC. For a topological space X, we set $C(X) := \{f | f : X \to \mathbb{R}, f \text{ is continuous}\}$.

Definition 2.2. Let K be an LSC with associated collection of similarities $\{F_i\}_{1 \leq i \leq N}$, and let $(\mathcal{E}, \mathcal{F})$ be a regular conservative irreducible symmetric Dirichlet form on $L^2(K, \mu)$. We call $(\mathcal{E}, \mathcal{F})$ a *standard self-similar Dirichlet form* on K if

$$\mathcal{F} \cap C(K) = \{f \in C(K) : \ f \circ F_i \in \mathcal{F} \text{ for any } i \in \{1, \ldots, N\}\}, \tag{2.4}$$

and there exists $\theta \in \mathbb{R}$ such that

$$\mathcal{E}(f,f) = \sum_{i=1}^{N} \rho_i^{-\theta} \mathcal{E}(f \circ F_i, f \circ F_i) \text{ for any } f \in \mathcal{F} \cap C(K). \tag{2.5}$$

We abbreviate such Dirichlet forms to SsDF and write $\mathcal{E}(f) := \mathcal{E}(f,f)$ for short in the later context.

Remark 2.3. It should be pointed out that a standard self-similar Dirichlet form $(\mathcal{E},\mathcal{F})$ on an LSC K is always strongly local. Indeed, $(\mathcal{E},\mathcal{F})$ is local by [13, Lemma 3.12] and the fact that $\{f\circ F_i : i\in\{1,\cdots,N\}\}\subset\mathcal{F}$ and the equation in (2.5) hold for any $f\in\mathcal{F}$ by (2.4), (2.5) and the denseness of $\mathcal{F}\cap C(K)$ in $(\mathcal{F},\mathcal{E}_1:=\mathcal{E}+\langle\cdot,\cdot\rangle_{L^2(K,\mu)})$, and thus $(\mathcal{E},\mathcal{F})$ is strongly local as a conservative local Dirichlet form.

One may choose other renormalization factors to replace (2.5) to define other self-similar Dirichlet forms. However, if the heat kernel estimate (2.3) is satisfied, then the renormalization factor can only be of the form $\rho_i^{-\theta}$. In fact, for the low dimensional setting $d_H<\beta$, it is well known that (2.3) implies that the resistance metric R associated with $(\mathcal{E},\mathcal{F})$ satisfies

$$c_5 d(x,y)^\theta\leq R(x,y)\leq c_6 d(x,y)^\theta,\quad \forall x,y\in K,$$

for $\theta=\beta-d_H$ and some positive constants c_5,c_6. See [17, Chapter 15], and also see [6] for a proof on the discrete setting.

3. The counter-example

In the following, we construct an LSC fractal K, associated with an IFS $\{F_i\}$ consisting of 104 contraction similarities. Let $a=\sqrt{\frac{7}{24}}-\frac{1}{2}\approx 0.0400617$, which is the positive solution of the equation

$$6(a^2+a)=\frac{1}{4}.$$

First, we define $F_i, 1\leq i\leq 13$ (see Figure 2 for an illustration of these mappings),

$$\begin{cases} F_{2j+1}(x)=ax+(\frac{j}{24},0), & \text{for } 0\leq j\leq 5,\\ F_{2j+2}(x)=a^2x+(a+\frac{j}{24},0), & \text{for } 0\leq j\leq 5,\\ F_{13}(x)=\frac{1}{4}x+(\frac{1}{4},0). \end{cases}$$

Next, we use symmetry to extend the above F_i to $1\leq i\leq 100$: let

$$F_i(x)=\Gamma_h\circ F_{27-i}\circ\Gamma_h(x),\quad 14\leq i\leq 26,$$

where Γ_h is the horizontal reflection in (2.1); let

$$F_{i+25j}(x)=\Gamma_{r_j}\circ F_i\circ\Gamma_{r_{4-j}}(x),\qquad 1\leq i\leq 25, 1\leq j\leq 3,$$

where Γ_{r_j}'s are the rotations in (2.2). Finally, we define the last 4 mappings,

$$F_{101}(x)=\frac{1}{4}x+(\frac{1}{4},\frac{1}{4}),\quad F_{102}(x)=\frac{1}{4}x+(\frac{1}{2},\frac{1}{4}),$$
$$F_{103}(x)=\frac{1}{4}x+(\frac{1}{2},\frac{1}{2}),\quad F_{104}(x)=\frac{1}{4}x+(\frac{1}{4},\frac{1}{2}).$$

Let K be the unique compact subset of $\mathbb{R}^2$ satisfying

$$K=\bigcup_{i=1}^{104}F_iK.$$

See Figure 3 for an illustration of the IFS and a sketch of K.

In the rest of this section, we will show the following result.

Theorem 3.1. *There is no SsDF on K.*

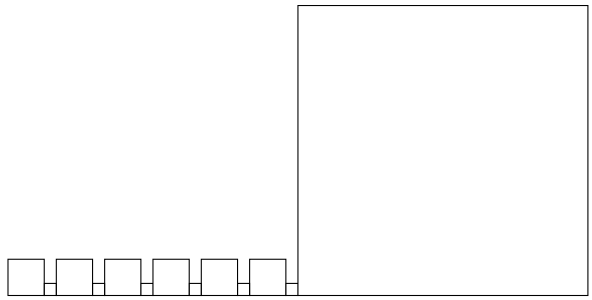

FIGURE 2. A sketch of mappings F_i, $1 \le i \le 13$ (not using the true value of a, which will make a^2 too small to be distinguished).

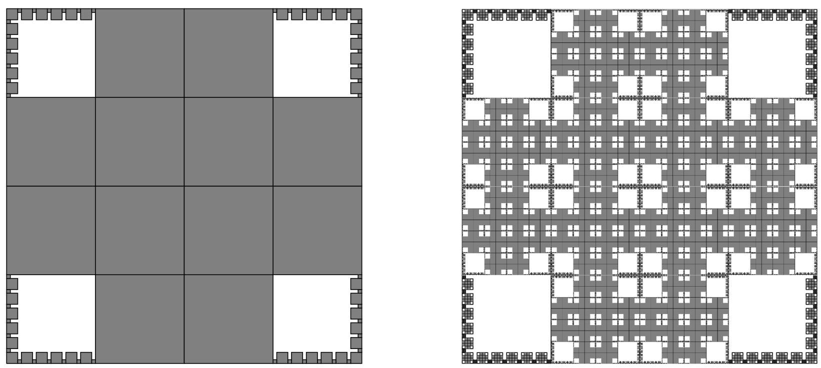

FIGURE 3. The IFS and a sketch of K.

We will prove Theorem 3.1 by contradiction. We assume that there is an SsDF on K satisfying (2.4) and (2.5) for some $\theta \in \mathbb{R}$. Then we will obtain an upper bound estimate $\theta \le \frac{1}{2}$ and a lower bound estimate $\theta > \frac{1}{2}$, which is of course impossible. Theorem 3.1 will then follow immediately.

Before proving Theorem 3.1, we present a few more concepts about capacity. Readers can find details in [8, 10].

Definition 3.2. Let X be a compact metrizable topological space, and μ be a Radon measure on X with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular conservative irreducible Dirichlet form on $L^2(X, \mu)$. We write

$$\mathcal{E}_1(f) = \mathcal{E}(f) + \|f\|^2_{L^2(X,\mu)},$$

for any $f \in \mathcal{F}$.

(a). For each open set $U \subset X$, we define the *capacity* of U by

$$\mathrm{Cap}(U) = \inf \left\{ \mathcal{E}_1(f) : f|_U \ge 1 \text{ a.e.}, \ f \in \mathcal{F} \right\}.$$

For a general set $B \subset X$, define the capacity of B by $\mathrm{Cap}(B) = \inf_{U \supset B} \mathrm{Cap}(U)$, where the infimum is taken over all open sets U containing B.

(b). For $A \subset X$ open, let $l(A)$ denote the collection of all real functions on A. A function $f \in l(A)$ is called *quasi-continuous* if for any $\varepsilon > 0$, there is open set $U \subset X$ such that $\mathrm{Cap}(U) < \varepsilon$ and $f|_{A\setminus U} \in C(A \setminus U)$.

For each $f \in \mathcal{F}$, we write $\tilde{f}$ for any quasi-continuous modification of f (whose existence is guaranteed by [10, Theorem 2.1.3]). In addition, write

$$\tilde{\mathcal{F}} = \{\tilde{f} : f \in \mathcal{F}\}.$$

We say $\tilde{f} = \tilde{g}$ if $\{x : f(x) \neq g(x)\}$ is of zero capacity. Note that by [10, Lemma 2.1.4], for $f, g \in \mathcal{F}$, $\tilde{f} = \tilde{g}$ if and only if $f = g$ (μ-a.e. on X).

(c). Let A, B be two disjoint closed subsets of X with positive capacity, we denote

$$R_{\mathcal{E}}(A,B) := \sup\{\frac{1}{\mathcal{E}(\tilde{f})} : \tilde{f} \in \tilde{\mathcal{F}}, \tilde{f}|_A = 0, \tilde{f}|_B = 1\}.$$

Write $R(A,B) = R_{\mathcal{E}}(A,B)$ for short if no confusion is caused, and set $R^{-1}(A,B) := R(A,B)^{-1}$.

Lemma 3.3. *Let X, μ and $(\mathcal{E},\mathcal{F})$ be as defined in Definition 3.2. Let A, B be two disjoint closed subsets of X with positive capacity. Then, $0 < R(A,B) < \infty$. In addition, for any $\varepsilon > 0$, there is $f \in \mathcal{F} \cap C(X)$ such that $f|_A = 0, f|_B = 1$ and $\mathcal{E}(f) \leq \varepsilon + R^{-1}(A,B)$.*

Proof. First, we show that $R(A,B) < \infty$. Assume that $R(A,B) = \infty$, then there is a sequence $\tilde{f}_n \in \tilde{\mathcal{F}}$ such that $\tilde{f}_n|_A = 0$, $\tilde{f}_n|_B = 1$ and $\lim\limits_{n\to\infty} \mathcal{E}(\tilde{f}_n) = 0$. In addition, we can assume that $0 \leq \tilde{f}_n \leq 1$ by the Markov property of $(\mathcal{E},\mathcal{F})$. By taking the limit of the Cesàro mean of a subsequence of $\tilde{f}_n$ (see [8, Theorem A.4.1]), one can find some $\tilde{f} \in \tilde{\mathcal{F}}$ such that $\mathcal{E}(\tilde{f}) = 0$, $\tilde{f}|_A = 0$ and $\tilde{f}|_B = 1$. This contradicts the assumption that $(\mathcal{E},\mathcal{F})$ is irreducible conservative (see [8, Theorem 2.1.11], noticing that the form is recurrent automatically).

Next we show that $R^{-1}(A,B) = \inf\{\mathcal{E}(f) : f \in \mathcal{F} \cap C(X), f|_A = 0, f|_B = 1\}$. Choose a sequence $f_n \in \mathcal{F} \cap C(X)$ such that $0 \leq f_n \leq 1, f_n|_A = 0$, $f_n|_B = 1$ and $\lim\limits_{n\to\infty} \mathcal{E}(f_n) = \inf\{\mathcal{E}(f) : f \in \mathcal{F} \cap C(X), f|_A = 0, f|_B = 1\}$. By taking the Cesàro mean of a subsequence of f_n, we have a limit $f \in \mathcal{F}$ in the $\mathcal{E}_1$-norm sense. It is direct to check that $\mathcal{E}(f+v) \geq \mathcal{E}(f)$ for any $v \in \mathcal{F} \cap C(X)$ satisfying $v|_{A\cup B} = 0$. By using [10, Lemma 2.3.4], we can see that $\mathcal{E}(f) = R^{-1}(A,B) = \inf\{\mathcal{E}(\tilde{f}) : \tilde{f} \in \tilde{\mathcal{F}}, \tilde{f}|_A = 0, \tilde{f}|_B = 1\}$. This implies the second statement. □

Lemma 3.4. *Let $(\mathcal{E},\mathcal{F})$ be an SsDF on K. Let $L_i = \overline{q_i, q_{i+1}}$ for $i = 1,2,3,4$ with cyclic notation $q_5 = q_1$. Then, $\mathrm{Cap}(L_i) > 0$ for all $i = 1,2,3,4$.*

Proof. For convenience, we write $\partial_0 K = \bigcup_{i=1}^4 L_i$, and write $\partial_1 K = \bigcup_{i=1}^{104} F_i(\partial_0 K)$.

Let $(\Omega, \mathcal{M}, X_t, \mathbb{P}_x)$ be the Hunt process associated with $(\mathcal{E},\mathcal{F})$ and $L^2(K,\mu)$, and write $\dot{\sigma}_A = \inf\{t \geq 0 : X_t \in A\}$ for a subset $A \subset K$. Since $(\mathcal{E},\mathcal{F})$ is irreducible and local, $\mathbb{P}_x(\dot{\sigma}_{K\setminus F_1(K\setminus\partial_0 K)} < \infty) = \mathbb{P}_x(\dot{\sigma}_{F_1(\partial_0 K)} < \infty) > 0$ for quasi every $x \in F_1 K$ by [10, Theorem 4.7.1, Theorem 4.5.1]. Thus, $\mathrm{Cap}\,\big(F_1(\partial_0 K)\big) > 0$ by [10, Theorem 4.2.1].

Now we show that $\mathrm{Cap}(\partial_0 K) > 0$. With the previous paragraph, it suffices to show that

$$\mathrm{Cap}(\partial_0 K) \geq c \cdot \mathrm{Cap}(\partial_1 K),$$

for some $c > 0$. Since $\partial_0 K$ is compact, by [10, Lemma 2.2.7], for any small $\varepsilon > 0$ there is $f \in \mathcal{F} \cap C(K)$ such that $f|_{\partial_0 K} = 1$ and $\mathcal{E}_1(f) \leq \mathrm{Cap}(\partial_0 K) + \varepsilon$. Define $g \in C(K)$ by

$g\circ F_i(x)=f(x),\forall 1\leq i\leq N$. Then, by (2.4), we see that $g\in\mathcal{F}$ and

$$\mathrm{Cap}(\partial_1 K)\leq \mathcal{E}_1(g)=\sum_{i=1}^N \rho_i^{-\theta}\mathcal{E}(f)+\|f\|^2_{L^2(K,\mu)}\leq c_1\big(\mathrm{Cap}(\partial_0 K)+\varepsilon\big),$$

for some $c_1>0$ depending on θ. The desired estimate follows immediately by noticing that ε is arbitrary.

By symmetry and the subadditivity of Cap ([10, Theorem A.1.2]), for each $i=1,2,3,4$, $\mathrm{Cap}(L_i)\geq \frac{\mathrm{Cap}(\partial_0 K)}{4}>0$. □

Now, we return to the proof of Theorem 3.1.

Proof of Theorem 3.1. Assume there is an SsDF on K, which means that there is a regular irreducible conservative Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$ satisfying (2.4) and (2.5) with some $\theta\in\mathbb{R}$.

Claim 1. $\theta\leq\frac{1}{2}$.

Proof of Claim 1. We will consider $R(L_2,L_4)$, where as in Lemma 3.4, $L_i=\overline{q_i,q_{i+1}}$. By Lemmas 3.3 and 3.4, we know that $0<R(L_2,L_4)<\infty$ and we can find $\tilde{h}\in\tilde{\mathcal{F}}$ such that $\tilde{h}|_{L_2}=0,\tilde{h}|_{L_4}=1$ and $\mathcal{E}(\tilde{h})=R^{-1}(L_2,L_4)$. By the irreducibility of $(\mathcal{E},\mathcal{F})$ and [8, Theorem 2.1.11], $\tilde{h}$ is unique, so that $\tilde{h}\circ\Gamma_v=\tilde{h}$ and $\tilde{h}\circ\Gamma_h=1-\tilde{h}$.

Let $S=\{38,39,88,89,101,102,103,104\}$ and $A_S=\bigcup_{i\in S}F_iK$. Note that $A_S=K\cap[0,1]\times[\frac{1}{4},\frac{3}{4}]$, i.e. the union of the 8 1-cells connecting L_2 to L_4 horizontally. Define $(\mathcal{E}_S,\mathcal{F}_S)$ as

$$\begin{aligned}\mathcal{F}_S=\big\{&f\in L^2(A_S,\mu|_{A_S}):f\circ F_i\in\mathcal{F},\forall i\in S,\\ &\text{and }\widetilde{f\circ F_i}|_{L_{i'}}=\widetilde{f\circ F_j}|_{L_{j'}}\circ F_j^{-1}\circ F_i|_{L_{i'}},\forall i,i',j,j'\text{ such that }F_iL_{i'}=F_jL_{j'}\big\},\end{aligned}$$

and

$$\mathcal{E}_S(f)=\sum_{i\in S}4^\theta\cdot\mathcal{E}(f\circ F_i).$$

Clearly, implied by (2.4) and (2.5), $\mathrm{Cap}(F_i(B))\geq\min\{\rho_i^{-\theta},\rho_i^{d_H}\}\,\mathrm{Cap}(B)$ for any $1\leq i\leq 104$ and any $B\subset K$. In particular, for any $f\in\mathcal{F}$ and any $1\leq i\leq 104$, $\tilde{f}\circ F_i$ is a quasi-continuous modification of $f\circ F_i\in\mathcal{F}$. So we have $\mathcal{F}|_{A_S}\subset\mathcal{F}_S$. Also, L_2,L_4 still have positive capacity with respect to $(\mathcal{E}_S,\mathcal{F}_S)$ on $L^2(A_S,\mu|_{A_S})$. By using symmetry, we can easily see that

$$R^{-1}_{\mathcal{E}_S}(L_2\cap A_S,L_4\cap A_S)=8\cdot4^{-2}\cdot4^\theta\mathcal{E}(\tilde{h}),$$

and the minimizer of the energy can be obtained by gluing scaled functions of $\frac{1}{4}\tilde{h}+\frac{k}{4}$, $k=0,1,2,3$ on F_iK, $i\in S$. So we have $\mathcal{E}(\tilde{h})\geq\mathcal{E}_S(\tilde{h}|_{A_S})\geq\frac{1}{2}\cdot4^\theta\mathcal{E}(\tilde{h})$. This implies that $\theta\leq\frac{1}{2}$.

Claim 2. $\theta\geq-\frac{\log 5}{\log a}$.

Proof of Claim 2. We will consider $R(F_1K,F_{26}K)$. Since F_1K and $F_{26}K$ have positive measures, both sets have positive capacity. For a small fixed $\varepsilon>0$, by Lemma 3.3, we can find $f\in\mathcal{F}\cap C(K)$ such that $f|_{F_1K}=0,f|_{F_{26}K}=1$ and $\mathcal{E}(f)\leq R^{-1}(F_1K,F_{26}K)+\varepsilon$. Now we use scaled copies of f to construct a function $g\in\mathcal{F}$ through the following 4 steps:

1. First, we construct g on $\bigcup_{i=1}^{12} F_iK$ as

$$g(x)=\begin{cases}0, & \text{if } x\in F_1K,\\ \frac{j-1}{10}, & \text{if } x\in F_{2j}K,\ \text{for } 1\leq j\leq 6,\\ \frac{1}{10}f\circ F_{2j+1}^{-1}(x)+\frac{j-1}{10}, & \text{if } x\in F_{2j+1}K,\ \text{for } 1\leq j\leq 5.\end{cases}$$

Note that the requirement, that the contraction ratio of $F_{2j}K$, $1\leq j\leq 5$ is the square of its neighboring cells, ensures the continuity of g.

2. Next, we extend g by symmetry to $\bigcup_{i=90}^{100} F_iK$. More precisely, let $g(x)=g\circ\Gamma_{d_1}(x)$ for any $x\in\bigcup_{i=90}^{100} F_iK$.

3. Then, we extend g to the left half of K by $\frac{1}{2}$, i.e. let $g(x)=\frac{1}{2}$ for any $x\in\Big(K\cap([0,\frac{1}{2}]\times[0,1])\Big)\setminus\Big((\bigcup_{i=1}^{12} F_iK)\cup(\bigcup_{i=90}^{100} F_iK)\Big)$.

4. Finally, we extend g to K so that $g-\frac{1}{2}$ is antisymmetric with respect to Γ_h, i.e. $g\circ\Gamma_h(x)+g(x)=1$ for any $x\in K$.

Now, we can easily see that $g\in\mathcal{F}\cap C(K)$, $g|_{F_1K}=0$, $g|_{F_{26}K}=1$ and

$$\mathcal{E}(g)=20\cdot(\frac{1}{10})^2\cdot a^{-\theta}\mathcal{E}(f)=\frac{1}{5}a^{-\theta}\mathcal{E}(f).$$

Since $\mathcal{E}(g)\geq R^{-1}(F_1K,F_{26}K)$, $\mathcal{E}(f)\leq R^{-1}(F_1K,F_{26}K)+\varepsilon$ and ε is arbitrary, we finally get $\frac{1}{5}a^{-\theta}\geq 1$, and thus $\theta\geq-\frac{\log 5}{\log a}$.

Finally, we can see there is a contradiction between Claim 1 and Claim 2, noticing that $-\frac{\log 5}{\log a}>\frac{1}{2}$. □

We conclude the paper with some remarks.

Remark 3.5. We should point out that Theorem 3.1 does not exclude the possibility that there exists a regular conservative irreducible symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$ satisfying (2.4) and (2.5) with some $(r_i)_{1\leq i\leq N}\in(0,\infty)^N$ in place of $(\rho_i^\theta)_{1\leq i\leq N}$. Indeed, such a situation occurs for the affine nested fractals introduced in [9] (see also [16, Section 3.8]), which are p.c.f. self-similar sets with good symmetry but allowing contraction ratios to be distinct like the LSC setting considered in this paper.

Remark 3.6. Our construction is partially inspired by the celebrated work of Sabot [19] on p.c.f. self-similar sets [15, 16]. If one tries to reproduce the arguments in [7, Subsection 4.2] with reasonable modification, one can see that the effective resistances between corner vertices are comparable with the Poincare constants. This observation has motivated the authors to construct the example with corner points loosely connected with inner cells, so that $R_n\ll\lambda_n$. Nevertheless, as pointed out in the last remark, it is unclear whether there exists an irreducible conservative regular symmetric Dirichlet form on $L^2(K,\mu)$ satisfying (2.4) and (2.5) with some $(r_i)_{1\leq i\leq N}\in(0,\infty)^N$ in place of $(\rho_i^\theta)_{1\leq i\leq N}$. The authors believe that there exist standard self-similar Dirichlet forms on many other LSC.

Remark 3.7. Based on the same idea, the construction can be extended to their higher dimensional analogues. We still only need three different sizes of cells, and the key to the construction is that, by choosing small size of 1-cells near the border of each face, and by placing the cells suitably, we can get a lower bound $\theta\geq 3-d-\varepsilon$, where d is the dimension.

For the 2-dimensional case, one can simply suppose that $(\mathcal{E},\mathcal{F})$ is a resistance form [16], since we want the sub-Gaussian heat kernel estimates. However, we still keep the Dirichlet form setting in the paper so the arguments can be extended to higher dimensions.

4. From sub-Gaussian heat kernel estimates to SsDF

At the end of the paper, we show that there doesn't exist a Dirichlet form (*not a priori assumed to be self-similar*) satisfying the sub-Gaussian heat kernel estimates (2.3) on the LSC K defined in Section 3. In fact, this follows immediately from the following observation.

Theorem 4.1. *Let K be a LSC and assume that there is a regular conservative symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$ which has a heat kernel satisfying (2.3). Then there is a SsDF on K with the equality $\beta=\theta+d_H$ holding, where β is the walk dimension of (2.3), θ is the same as in (2.5) and d_H is the Hausdorff dimension of K.*

We will use the idea of [18, Section 6], which is modified in [7, Section 5], to construct a self-similar form as a subsequential limit generated from $(\mathcal{E},\mathcal{F})$. We need the following lemma inspired by [12, Theorem 4.2] by Grigor'yan, Hu and Lau. Throughout the following context, we always use $B(x,\rho)$ to denote the open ball centered at x with radius ρ in the Euclidean metric d for $x\in K$, $\rho>0$. It is well known (see [14, Theorem 5.3.1] for example) that there exist $c,c'\in(0,\infty)$ such that $c\rho^{d_H}\leq\mu(B(x,\rho))\leq c'\rho^{d_H}$ for any $x\in K$ and any $\rho\in(0,\mathrm{diam}K]$, where $\mathrm{diam}K:=\sup_{x,y\in K}d(x,y)$.

Lemma 4.2. *Assume the same settings as in Theorem 4.1. For each $f\in L^2(K,\mu)$ and $\rho>0$, we define*

$$I_\rho(f)=\rho^{-d_H-\beta}\int_K\int_{B(x,\rho)}\big(f(x)-f(y)\big)^2\mu(dy)\mu(dx).$$

Then $\mathcal{F}=\big\{f\in L^2(K,\mu):\sup_{0<\rho\leq1}I_\rho(f)<\infty\big\}=\big\{f\in L^2(K,\mu):\liminf_{\rho\to0}I_\rho(f)<\infty\big\}$ and

$$C_1\sup_{0<\rho\leq1}I_\rho(f)\leq\mathcal{E}(f)\leq C_2\liminf_{\rho\to0}I_\rho(f),\ \text{for any } f\in\mathcal{F}\tag{4.1}$$

for some positive finite constants C_1,C_2.

Proof. For any $t>0$, let $\mathcal{E}_t$ be a quadratic form on $L^2(K,\mu)$ defined by

$$\mathcal{E}_t(f)=\frac{1}{2t}\int_K\int_K\big(f(x)-f(y)\big)^2p(t,x,y)\mu(dx)\mu(dy),$$

where $p(t,x,y)$ is the heat kernel associated with $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$. Then, by the argument in [12, Section 4.2], we know that for any $f\in L^2(K,\mu)$, $\mathcal{E}_t(f)$ is decreasing in t. Moreover, $f\in\mathcal{F}$ if and only if $\lim_{t\to0}\mathcal{E}_t(f)<\infty$, and in this case $\mathcal{E}(f)=\lim_{t\to0}\mathcal{E}_t(f)$.

Then, by applying the lower bound estimate of (2.3), for any $0<t\leq(2\mathrm{diam}K)^\beta$, $f\in L^2(K,\mu)$,

$$\mathcal{E}_t(f)\geq\frac{1}{2t}\int_K\int_{B(x,t^{1/\beta})}\big(f(x)-f(y)\big)^2p(t,x,y)\mu(dy)\mu(dx)\geq C_3I_{t^{1/\beta}}(f)\tag{4.2}$$

for some positive finite constant C_3 depending only on (2.3). Conversely, by the upper bound estimate of (2.3), for each $0 < t \leq 1$ and $0 \leq m \leq \eta(t) := \max\big(\mathbb{Z}\cap[0,\log_2(\mathrm{diam}K/t^{1/\beta})]\big)$, we have

$$\begin{aligned}\mathcal{E}_t(f) =&\frac{1}{2t}\int_K\int_{B(x,2^m t^{1/\beta})}\big(f(x)-f(y)\big)^2 p(t,x,y)\mu(dy)\mu(dx)+\\ &\sum_{k=m}^{\eta(t)}\frac{1}{2t}\int_K\int_{B(x,2^{k+1}t^{1/\beta})\setminus B(x,2^k t^{1/\beta})}\big(f(x)-f(y)\big)^2 p(t,x,y)\mu(dy)\mu(dx)\\ \leq& C_{4,m}I_{2^m t^{1/\beta}}(f)+C_5\sum_{k=m}^{\eta(t)}2^{k(d_H+\beta)}e^{-c_4\cdot 2^{\frac{k\beta}{\beta-1}}}\cdot I_{2^{k+1}t^{1/\beta}}(f),\end{aligned}\tag{4.3}$$

where $C_{4,m}$ is a positive finite constant depending only on m and (2.3), c_4 is the same as in (2.3), and C_5 is a positive finite constant depending only on (2.3).

By combining (4.2) and (4.3), it follows that $\mathcal{F}=\{f\in L^2(K,\mu):\sup_{0<\rho\leq 1}I_\rho(f)<\infty\}$. In addition, the left side inequality of (4.1) is immediate from (4.2). To see the right side of (4.1), we take m large enough in (4.3) so that $C_3^{-1}C_5\sum_{k=m}^{\infty}2^{k(d_H+\beta)}e^{-c_4\cdot 2^{\frac{k\beta}{\beta-1}}}\leq\frac{1}{2}$, and then for any $0<t\leq(2^{-m}\mathrm{diam}K)^\beta$,

$$\mathcal{E}_t(f)\leq C_{4,m}I_{2^m t^{1/\beta}}(f)+\frac{1}{2}\mathcal{E}_t(f)$$

by using (4.3), (4.2) and the monotonicity of $s\mapsto\mathcal{E}_s(f)$. Hence by letting $t\to 0$ we obtain $\mathcal{F}=\{f\in L^2(K,\mu):\liminf_{\rho\to 0}I_\rho(f)<\infty\}$ and $\mathcal{E}(f)\leq 2C_{4,m}\liminf_{\rho\to 0}I_\rho(f)$ for any $f\in\mathcal{F}$. □

We also refer to some other facts from [12] for further discussions. First, by [12, Theorem 4.8], we have $\beta\geq 2>d_H$. Hence, by [12, Theorems 4.2 and 4.11], we have $\mathcal{F}\subset C(K)$, so we can avoid the inconvenience of having to treat noncontinuous functions.

Proof of Theorem 4.1. For $n\geq 0$, define

$$\bar{\mathcal{E}}_n(f)=\sum_{w\in W_n}\rho_w^{-\theta}\mathcal{E}(f\circ F_w),\quad\forall f\in\mathcal{F}_n,$$

where $\theta:=\beta-d_H$ and $\mathcal{F}_n:=\{f\in C(K):f\circ F_w\in\mathcal{F},\forall w\in W_n\}$. We need to show $\mathcal{F}_n=\mathcal{F}$ and $C_1\mathcal{E}(f)\leq\bar{\mathcal{E}}_n(f)\leq C_2\mathcal{E}(f)$ for some positive finite constants C_1,C_2 independent of n. For convenience, we write $\mathcal{E}(f)=\infty$ if $f\notin\mathcal{F}$ (similarly $\bar{\mathcal{E}}_n(f)=\infty$ if $f\notin\mathcal{F}_n$).

We first show $\bar{\mathcal{E}}_n(f)\leq C_2\mathcal{E}(f)$. For any small ρ, we have

$$I_\rho(f)\geq\sum_{w\in W_n}\rho^{-d_H-\beta}\int_{F_wK}\int_{F_wK\cap B(x,\rho)}\big(f(x)-f(y)\big)^2\mu(dy)\mu(dx)=\sum_{w\in W_n}\rho_w^{-\theta}I_{\rho/\rho_w}(f\circ F_w).$$

The inequality $\bar{\mathcal{E}}_n(f)\leq C_2\mathcal{E}(f)$ follows immediately by Lemma 4.2 and letting $\rho\to 0$. It is also clear that $\mathcal{F}\subset\mathcal{F}_n$.

The other inequality $\bar{\mathcal{E}}_n(f) \geq C_1\mathcal{E}(f)$ needs more care. For $f \in \mathcal{F}_n$, we need to estimate $I_\rho(f)$ (for small ρ) across the cells:

$$I_\rho(f) - \sum_{w\in W_n} \rho_w^{-\theta} I_{\rho/\rho_w}(f\circ F_w) = \sum_{w\neq w'\in W_n} I_{\rho,w,w'}(f), \tag{4.4}$$

where

$$I_{\rho,w,w'}(f) = \rho^{-d_H-\beta}\int\int_{\{(x,y)\in F_wK\times F_{w'}K: d(x,y)<\rho\}} \big(f(x)-f(y)\big)^2\mu(dx)\mu(dy).$$

For convenience, we fix any pair $(w,w') \in W_n \times W_n$ with $F_wK \cap F_{w'}K \neq \emptyset$, and let $L_{w,w'} = F_wK \cap F_{w'}K$.

We now consider two possible cases.

Firstly, assume that $L_{w,w'} = \{z_{w,w'}\}$ is a single point. We choose large enough $C_3 > 1$ and let $\rho' = C_3\rho$. For $k \geq 0$, we set

$$\begin{aligned}
I_{\rho,w,w',k} &:= \rho^{d_H-\beta}\fint_{F_wK\cap B(z_{w,w'},2^{-k}\rho')}\fint_{F_{w'}K\cap B(z_{w,w'},2^{-k}\rho')}\big(f(x)-f(y)\big)^2\mu(dx)\mu(dy),\\
D_{\rho,w,w',k} &:= (2^{-k}\rho)^{d_H-\beta}\fint_{F_wK\cap B(z_{w,w'},2^{-k}\rho')}\fint_{F_wK\cap B(z_{w,w'},2^{-k+1}\rho')}\big(f(x)-f(x')\big)^2\mu(dx)\mu(dx'),
\end{aligned}$$

where for convenience, we set $\fint_A f(x)\mu(dx) = \frac{1}{\mu(A)}\int_A f(x)\mu(dx)$. See (4.7) for an explanation of $I_{\rho,w,w',k}$ and $D_{\rho,w,w',k}$. In this case, by denoting $A_{w,k} = F_wK\cap B(z_{w,w'},2^{-k}\rho')$ and $A_{w',k} = F_{w'}K\cap B(z_{w,w'},2^{-k}\rho')$ for short, and by using the Minkowski inequality, we can check that

$$\begin{aligned}
&\sqrt{I_{\rho,w,w',k}}\\
=&\sqrt{\rho^{d_H-\beta}\fint_{A_{w,k}}\fint_{A_{w',k}}\fint_{A_{w,k+1}}\fint_{A_{w',k+1}} \Big(\big(f(x)-f(x')\big)+\big(f(x')-f(y')\big)+\big(f(y')-f(y)\big)\Big)^2\mu(dy')\mu(dx')\mu(dy)\mu(dx)}\\
\leq& 2^{-(k+1)\theta/2}\sqrt{D_{\rho,w,w',k+1}} + \sqrt{I_{\rho,w,w',k+1}} + 2^{-(k+1)\theta/2}\sqrt{D_{\rho,w',w,k+1}}.
\end{aligned}$$

Summing up the above inequality over k and noticing that f is continuous on K, we get

$$\sqrt{I_{\rho,w,w',0}} \leq \sum_{k=1}^\infty 2^{-k\theta/2}\big(\sqrt{D_{\rho,w,w',k}} + \sqrt{D_{\rho,w',w,k}}\big).$$

Then by using the Cauchy-Schwarz inequality, we can see that

$$I_{\rho,w,w',0} \leq \big(2\sum_{k=1}^\infty 2^{-k\theta/2}\big)\cdot\big(\sum_{k=1}^\infty 2^{-k\theta/2}D_{\rho,w,w',k} + \sum_{k=1}^\infty 2^{-k\theta/2}D_{\rho,w',w,k}\big). \tag{4.5}$$

Secondly, we assume that $L_{w,w'}$ is a line segment. We let ν be the Lebesgue measure (length) on $L_{w,w'}$, and we still choose large enough $C_3 > 1$ and let $\rho' = C_3\rho$. Define for

$k \geq 0$,

$$I_{\rho,w,w',k} := \rho^{d_H-\beta} \int_{L_{w,w'}} \frac{\nu(dz)}{\rho} \fint_{F_wK\cap B(z,2^{-k}\rho')} \fint_{F_{w'}K\cap B(z,2^{-k}\rho')} \big(f(x)-f(y)\big)^2 \mu(dx)\mu(dy),$$

$$D_{\rho,w,w',k} := (2^{-k}\rho)^{d_H-\beta} \int_{L_{w,w'}} \frac{\nu(dz)}{2^{-k}\rho} \fint_{F_wK\cap B(z,2^{-k}\rho')} \fint_{F_wK\cap B(z,2^{-k+1}\rho')} \big(f(x)-f(x')\big)^2 \mu(dx)\mu(dx'),$$

Here we need to explain the reason that we choose the measure $\nu(dz)/\rho$ (or $\nu(dz)/(2^{-k}\rho)$) in the integral: for any x, y close enough, and when ρ is small enough, we have $\nu\big(\{z : x \in B(z,\rho'), y \in B(z,\rho')\}\big) \approx \rho$. In this case, similarly as above, we can check that

$$\sqrt{I_{\rho,w,w',0}} \leq \sum_{k=1}^{\infty} 2^{-k(1+\theta)/2}\big(\sqrt{D_{\rho,w,w',k}} + \sqrt{D_{\rho,w',w,k}}\big),$$

and then

$$I_{\rho,w,w',0} \leq \big(2\sum_{k=1}^{\infty} 2^{-k(\theta+1)/2}\big) \cdot \big(\sum_{k=1}^{\infty} 2^{-k(\theta+1)/2} D_{\rho,w,w',k} + \sum_{k=1}^{\infty} 2^{-k(\theta+1)/2} D_{\rho,w',w,k}\big). \tag{4.6}$$

In both cases, if $C_3 > 1$ is chosen suitably, we have for any ρ small enough (much smaller than the lengths of any line segments $L_{w,w'}$),

$$\begin{cases} I_{\rho,w,w',0} \geq C_4 I_{\rho,w,w'}(f), \\ \sum\limits_{w'\in W_n\setminus\{w\}, F_{w'}K\cap F_wK\neq\emptyset} D_{\rho,w,w',k} \leq C_4 \cdot \rho_w^{-\theta} I_{2^{-k+2}\rho'/\rho_w}(f\circ F_w), \end{cases} \tag{4.7}$$

where C_4 is a constant depending only on the fractal K and on (2.3). By combining (4.5), (4.6) and (4.7), and by Lemma 4.2, we get for any ρ small enough

$$\sum_{w\neq w'\in W_n} I_{\rho,w,w'} \leq C_5 \sum_{w\in W_n} \rho_w^{-\theta} \mathcal{E}(f\circ F_w),$$

for some $C_5 > 0$ depending only on the fractal K and (2.3). Hence, by (4.4), combining Lemma 4.2, we finally see

$$\mathcal{E}(f) \leq C_6 \bar{\mathcal{E}}_n(f)$$

for some $C_6 > 0$. In addition, it is clear that $\mathcal{F}_n \subset \mathcal{F}$.

To conclude, we have proved that $\mathcal{F}_n = \mathcal{F}$ and that there are C_1, C_2 independent of n, f such that $C_1\mathcal{E}(f) \leq \bar{\mathcal{E}}_n(f) \leq C_2\mathcal{E}(f)$. We proceed to finish the construction following the idea of Kusuoka-Zhou [18]. Let $\hat{\mathcal{F}}$ be a $\mathbb{Q}$-vector subspace of $\mathcal{F}$ with countably many elements which is dense in $\mathcal{F}$ with respect to the norm $\|f\|_{\mathcal{E}_1} := \sqrt{\mathcal{E}(f) + \|f\|^2_{L^2(K,\mu)}}$. To achieve this, one can simply choose a dense $\mathbb{Q}$-vector subspace H of $L^2(K,\mu)$ with countably many elements, and let $\hat{\mathcal{F}} = U_1(H)$, where U_1 is the resolvent operator associated with $\mathcal{E}_1$, i.e. $\mathcal{E}_1(U_1 f, g) = \int_K fg d\mu$, for any $f \in L^2(K,\mu)$ and $g \in \mathcal{F}$. Then by a diagonal argument, there is a subsequence $\{n_l\}_{l\geq 1}$ such that the limit

$$\bar{\mathcal{E}}(f) := \lim_{l\to\infty} \frac{1}{n_l} \sum_{m=1}^{n_l} \bar{\mathcal{E}}_m(f)$$

exists for any $f \in \hat{\mathcal{F}}$. Furthermore, for the general case that $f \in \mathcal{F}$, we can also prove that the limit $\bar{\mathcal{E}}(f) := \lim_{l\to\infty} \frac{1}{n_l}\sum_{m=1}^{n_l} \bar{\mathcal{E}}_m(f)$ exists. Indeed, for any $\varepsilon > 0$, we choose $g \in \hat{\mathcal{F}}$ such that $\|f-g\|_{\mathcal{E}_1} < \varepsilon$. Then, by using the inequality $\Big|\sqrt{\frac{1}{n_l}\sum_{m=1}^{n_l}\bar{\mathcal{E}}_m(f)} - \sqrt{\frac{1}{n_l}\sum_{m=1}^{n_l}\bar{\mathcal{E}}_m(g)}\Big|^2 \le \frac{1}{n_l}\sum_{m=1}^{n_l}\bar{\mathcal{E}}_m(f-g) \le C_2\mathcal{E}(f-g), \forall l \ge 1$, we have

$$
\begin{aligned}
&\limsup_{l,l'\to\infty}\left|\sqrt{\frac{1}{n_l}\sum_{m=1}^{n_l}\bar{\mathcal{E}}_m(f)} - \sqrt{\frac{1}{n_{l'}}\sum_{m=1}^{n_{l'}}\bar{\mathcal{E}}_m(f)}\right| \\
\le&\limsup_{l,l'\to\infty}\left(\left|\sqrt{\frac{1}{n_l}\sum_{m=1}^{n_l}\bar{\mathcal{E}}_m(f)} - \sqrt{\frac{1}{n_l}\sum_{m=1}^{n_l}\bar{\mathcal{E}}_m(g)}\right| + \left|\sqrt{\frac{1}{n_l}\sum_{m=1}^{n_l}\bar{\mathcal{E}}_m(g)} - \sqrt{\frac{1}{n_{l'}}\sum_{m=1}^{n_{l'}}\bar{\mathcal{E}}_m(g)}\right|\right. \\
&\qquad\left. + \left|\sqrt{\frac{1}{n_{l'}}\sum_{m=1}^{n_{l'}}\bar{\mathcal{E}}_m(g)} - \sqrt{\frac{1}{n_{l'}}\sum_{m=1}^{n_{l'}}\bar{\mathcal{E}}_m(f)}\right|\right) \\
\le&2\sqrt{C_2}\varepsilon + \limsup_{l,l'\to 0}\left|\sqrt{\frac{1}{n_l}\sum_{m=1}^{n_l}\bar{\mathcal{E}}_m(g)} - \sqrt{\frac{1}{n_{l'}}\sum_{m=1}^{n_{l'}}\bar{\mathcal{E}}_m(g)}\right| = 2\sqrt{C_2}\varepsilon.
\end{aligned}
$$

This implies that $\sqrt{\frac{1}{n_l}\sum_{m=1}^{n_l}\bar{\mathcal{E}}_m(f)}, l \ge 1$ is a Cauchy sequence, and so its limit exists. Finally, by the inequality $C_1\mathcal{E}(f) \le \bar{\mathcal{E}}_m(f) \le C_2\mathcal{E}(f), \forall m \ge 1, f \in \mathcal{F}$, we know that

$$C_1 \cdot \mathcal{E}(f) \le \bar{\mathcal{E}}(f) \le C_2 \cdot \mathcal{E}(f), \quad \forall f \in \mathcal{F}. \tag{4.8}$$

Immediately, the functional $\bar{\mathcal{E}}$ induces a regular symmetric Dirichlet form $(\bar{\mathcal{E}}, \mathcal{F})$ on $L^2(K,\mu)$ which is conservative, i.e., satisfies $\bar{\mathcal{E}}(\mathbf{1}) = 0$. The self-similarity (2.4) of the domain $\mathcal{F}$ follows from the fact $\mathcal{F} = \mathcal{F}_1$, and the self-similarity (2.5) of $\bar{\mathcal{E}}$ is immediate from the construction. The irreducibility of $(\bar{\mathcal{E}}, \mathcal{F})$ follows from Lemma 4.2, (4.8) and [8, Theorem 2.1.11]. □

Remark 4.3. In his recent survey on analysis on fractal spaces and heat kernels, Grigor'yan [11] posed an open question that *on a given Ahlfors d_H-regular metric measure space (M, d, μ), when and how one can construct a strongly local regular symmetric Dirichlet form on $L^2(M, \mu)$ with the heat kernel estimates (2.3)*. In view of Theorem 4.1, the example constructed in Section 3 may serve as a non-existence example.

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