HAUSDORFF AND BOX DIMENSION OF SELF-AFFINE SET IN LOCALLY COMPACT NON-ARCHIMEDEAN FIELD

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ABSTRACT. In this paper we consider affine iterated function systems in locally compact non-Archimedean field $\mathbb{F}$. We establish the theory of singular value composition in $\mathbb{F}$ and compute box and Hausdorff dimension of self-affine set in $\mathbb{F}^n$, in generic sense, which is an analogy of Falconer's result for real case. The result has the advantage that no additional assumptions needed to be imposed on the norms of linear parts of affine transformation while such norms are strictly less than $\frac{1}{2}$ for real case, which benefits from the non-Archimedean metric on $\mathbb{F}$.

1. INTRODUCTION

For a self-affine set in $\mathbb{R}^n$, Falconer [9] introduced singular value function of linear transformations in its associated iterated function system to determine its Hausdorff dimension. Precisely, given $M \geq 2$, let $T_1, T_2, \cdots, T_M$ be a set of contractive, non-singular linear transformations on $\mathbb{R}^n$, and $\mathbf{b} = (b_1, b_2, \cdots, b_M)$ be a vector of M points in $\mathbb{R}^n$. We denote by $K(\mathbf{b})$ the self-affine set associated with the affine function system $\{S_1, S_2, \cdots, S_M\}$ of the form $S_i(x) = T_i(x) + b_i$, i.e., $K(\mathbf{b})$ is the unique non-empty compact set satisfying

$$K(\mathbf{b}) = \bigcup_{i=1}^{M} S_i(K(\mathbf{b})).$$

Providing that $\|T_i\| < \frac{1}{2}$ for each i, where $\|T_i\|$ is the operator norm of T_i. Then, there is a number $d(T_1, T_2 \cdots, T_M)$ depending on the linear transformations $T_1, T_2, \cdots, T_M$, such that the invariant set $K(\mathbf{b})$ has Hausdorff dimension $\min\{n, d(T_1, T_2, \cdots, T_M)\}$ for almost all $\mathbf{b} \in \mathbb{R}^{nM}$ in the sense of nM-dimensional Lebesgue measure.

The critical number $d(T_1, T_2, \cdots, T_M)$ is given in terms of singular values of linear transformations, which is defined as follows. If T is a non-singular linear transformation on $\mathbb{R}^n$, the singular values of T are positive square roots of eigenvalues of T^tT, written as $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, where T^t is the transpose of T. They can be viewed as the length of the (mutually perpendicular) principle semiaxes of $T(B)$, where B is the unit ball in $\mathbb{R}^n$. The singular value function ϕ^s is defined by

$$\phi^s(T) = \alpha_1 \alpha_2 \cdots \alpha_{m-1} \alpha_m^{s-m+1},$$

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for $0 \le s \le n$, where m is the smallest integer greater than or equal to s, and

$$\phi^s(T) = (\alpha_1\alpha_2\cdots\alpha_n)^{\frac{s}{n}},$$

for $s > n$. It is easy to prove that for each $s \ge 0$, ϕ^s is submultiplicative, i.e.

$$\phi^s(TU) \le \phi^s(T)\phi^s(U),$$

for any $T, U \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, where $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ represents all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. The critical exponent $d(T_1, T_2, \cdots, T_M)$ is defined to be the unique non-negative solution s to equation

$$P(\phi^s) := \lim_{k\to+\infty} \frac{1}{k} \log\left(\sum_{w_1,\cdots,w_k} \phi^s(T_{w_1}T_{w_2}\cdots T_{w_k})\right) = 0, \tag{1.1}$$

where the sum is taken over all finite words $(w_1w_2\cdots w_m)$ of length m with $1 \le w_i \le M$, and the number $P(\phi^s)$ in (1.1) is called the topological pressure of ϕ^s. The submultiplicativity of ϕ^s ensures convergence of limit in (1.1). Moreover, as a function of s, the $P(\phi^s)$ is continuous and strictly decreasing, and it is greater than 0 when $s = 0$ and is less than 0 for some large s. Hence there exists a unique s for which the limit equals to 0.

Falconer [9] proved the dimension result indicated above under the assumption that $\|T_i\| < \frac{1}{3}$ for each i, using potential-theoretic methods. Later, Solomyak [21] pointed out that $\|T_i\| < \frac{1}{3}$ could be weakened to $\|T_i\| < \frac{1}{2}$. The constant $\frac{1}{2}$ is proved to be sharp by Edgar in [5]. Recently, the result was extended to self-affine sets in the sub-Riemannian metric setting of the Heisenberg group by Balogh and Tyson [3].

In this paper, we consider the Haussdorff dimension of self-affine set in non-Archimedean field, and let $\mathbb{F}$ be a non-Archimedean field, see the next section for its defintion and related properties. Consider the same type affine iterated function system $\{T_1x + b_1, T_2x + b_2, \cdots, T_Mx + b_M\}$ where T_i are non-singular contractive linear transformations on $\mathbb{F}^n$ and b_i are points in $\mathbb{F}^n$, let μ be the Haar measure on $\mathbb{F}$, and μ^{nM} is the nM-dimensional product measure generated by μ. Similar to the case on $\mathbb{R}^n$, we use $\mathbf{b}$ to denote the vector $(b_1, b_2\cdots, b_M) \in \mathbb{F}^{nM}$ and $K(\mathbf{b})$ to denote the associated self-affine set. We will define the similar critical exponent $d(T_1, T_2, \cdots, T_M)$ (still denoted by $d(T_1, T_2, \cdots, T_M)$ without causing any confusion) by introducing the analogous concepts of singular value composition and singular value function in non-Archimedean field, and then prove the following analog of the Falconer's formula under additional condition that $\mathbb{F}$ is a locally compact field. Similarly, we use $\dim_H$ to denote the Hausdorff dimension and use $\dim_B$ to denote the box dimension.

Theorem 1.1. *Let $\mathbb{F}$ be a locally compact non-Archimedean field, T_i be non-singular contractive linear transformations on $\mathbb{F}^n$ for any $1 \le i \le M$, and $\boldsymbol{b} = (b_1, b_2\cdots, b_M) \in \mathbb{F}^{nM}$. Then, we have*
(i) $\dim_H K(\boldsymbol{b}) \le \overline{\dim}_B K(\boldsymbol{b}) \le d(T_1, T_2, \cdots, T_M)$ *for all* $\boldsymbol{b} \in \mathbb{F}^{nM}$;
(ii) $\dim_H K(\boldsymbol{b}) = \dim_B K(\boldsymbol{b}) = \min\{n, d(T_1, T_2, \cdots, T_M)\}$ *for* μ^{nM}*-a.e.* $\boldsymbol{b} \in \mathbb{F}^{nM}$.

In the non-Archimedean field $\mathbb{F}$, we still denote the operator norm of a linear transformations A with $\|A\|$. Then, the results no longer need $\|T_i\| < \frac{1}{2}$ for any $i \le M$. Actually, from the discreteness of $\|\cdot\|$ in $\mathbb{F}$, $\|T_i\| < 1$ means $\|T_i\| \le \frac{1}{2}$. In particular, when $q = 2$, Theorem 1.1 still holds in the case that some $\|T_i\|$ may reach the critical value $\frac{1}{2}$, which is forbidden in the real numbers case.

2. LOCALLY COMPACT NON-ARCHIMEDEAN FIELD

Let $\mathbb{F}$ be an infinite field in which a non-Archimedean valuation is defined for all $x \in \mathbb{F}$, i.e. a map from $\mathbb{F}$ to the nonnegative real numbers such that, for any x, $y \in \mathbb{F}$
(i) $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|xy\| = \|x\| \cdot \|y\|$,
(iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$.
Denote $O = \{a \in \mathbb{F} : \|a\| \leq 1\}$, and $P = \{a \in \mathbb{F} : \|a\| < 1\}$. Then O forms a ring under addition and multiplication as defined in the field $\mathbb{F}$. Moreover the subset P forms maximal ideal in O since outside P there are only units. Clearly O/P is a field, we call it the residue class field of the filed $\mathbb{F}$.

Moreover, if $\mathbb{F}$ is a locally compact non-Archimedean field, i.e. for any $x \in \mathbb{F}$, there exists a compact set C containing a neighbourhood of x. We have following Lemma.

Lemma 2.1. *Let $(\mathbb{F}, \|\cdot\|)$ be a locally compact non-Archimedean field, and d be the nature metric induced by $\|\cdot\|$, then*
(i)$(\mathbb{F}, d)$ *is complete, and O is compact.*
(ii)O/P *is a finite set.*
(iii)P *is a principal ideal, i.e. there exists $\pi \in P$ s.t. $P = \pi O$.*

Denote $\mathrm{card}(O/P) = q$. Usually, we let $\|\pi\| = \frac{1}{q}$, it is a normalization of $\|\cdot\|$. At this point, we have following propostion.

Proposition 2.1. *Let μ be the Haar measure on $\mathbb{F}$, here $\mathbb{F}$ is regarded as a compact group under "+", such that $\mu(O) = 1$. Then, for any $A \in \mathcal{B}(\mathbb{F})$ and any $c \in \mathbb{F}$, $\mu(cA) = \| c \| \cdot \mu(A)$, where $\mathcal{B}(\mathbb{F})$ is the Borel algebra in $\mathbb{F}$.*

Propostion 2.1 and Lemma 2.1 are the basic results of algebra, which are proved in [4]. Next, we will briefly give two classical examples of locally compact non-Archimedean field, see [13, 22] for more details.

Example 2.1. *Let $\mathbb{K}$ be a finite filed with q elements, and $\mathbb{K}((X^{-1}))$ be the field of formal Laurent series, i.e.*

$$\mathbb{K}((X^{-1})) = \left\{ \sum_{n=n_0}^{+\infty} x_n X^{-n} : x_n \in \mathbb{K} \text{ and } n_0 \in \mathbb{Z} \right\}.$$

Denote $\deg(x) = -\inf\{n \in \mathbb{Z} : x_n \neq 0\}$, where $x = \sum_{n=n_0}^{+\infty} x_n X^{-n} \in \mathbb{K}((X^{-1}))$. In particular, $\deg(0) = -\infty$. Define the norm of x to be $\|x\| = q^{\deg(x)}$, with $\|0\| = 0$, then we have following:
(1) $\|x\| = 0$ *if and only if* $x = 0$;
(2) $\|xy\| = \|x\| \cdot \|y\|$;
(3) $\|\alpha x + \beta y\| \leq \max(\|x\|, \|y\|)$ (for any α, $\beta \in \mathbb{K}$);
(4) *For any $\alpha, \beta \in \mathbb{K}$, $\alpha \neq 0$, $\beta \neq 0$, if $\|x\| \neq \|y\|$, then $\|\alpha x + \beta y\| = \max(\|x\|, \|y\|)$.*
Notice, $P = \{x \in \mathbb{K}((X^{-1})) : \|x\| < 1\}$ is isomorphic to $\prod_{n \geq 1} \mathbb{K}$, which is compact. Thus $\mathbb{K}((X^{-1}))$ is a locally compact non-Archimedean field.

Example 2.2. *Let p be a prime number, and $\mathbb{Q}_p$ be the p-adic field. Then for any $x \in \mathbb{Q}_p$, we have $x = \sum_{n=n_0}^{+\infty} a_n p^n$, where $a_n \in \{0, 1, \cdots, p-1\}$. We denote $ord(x) = \inf\{n \in \mathbb{Z} : a_n \neq 0\}$. In particular, $ord(0) = \infty$. And define the norm* *of x to be $|x|_p = p^{-ord(x)}$, then $|\cdot|_p$ is a non-Archimedean norm. Same, we can get $P = \{x \in \mathbb{Q}_p : |x|_p < 1\}$ is compact, moreover, $(\mathbb{Q}_p, |\cdot|_p)$ is a locally compact non-Archimedean field.*

Now, let k be an integer and $\mathbb{F}^k = \mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}$ be the k-dimensional vector space over $\mathbb{F}$. The norms on $\mathbb{F}^k$ is $\|\cdot\| : \mathbb{F}^k \to [0, +\infty)$ with $\|x\| = \max_{1\leq i\leq k} \|x_i\|$, where $x = (x_1, x_2, \cdots, x_k) \in \mathbb{F}^k$, this norm is also a non-Archimedean one as $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for any $x, y \in \mathbb{F}^k$. Without causing misunderstanding, we still let d on $\mathbb{F}^k$ be the nature metric induced by $\|\cdot\|$. Then it is easy to check $(\mathbb{F}^k, d)$ is complete and locally compact. Besides, let $m_1 = \mu \times \mu \cdots \times \mu$ be the product measure on $\mathbb{F}^k$, and m_2 be the Haar measure on $\mathbb{F}^k$. Then for any set $A \in \mathcal{B}(\mathbb{F}) \times \mathcal{B}(\mathbb{F}) \cdots \times \mathcal{B}(\mathbb{F})$, both of m_1 and m_2 are satisfied $m_i(x + A) = m_i(A)$, $i = 1, 2$. Notice $\mathcal{B}(\mathbb{F}) \times \mathcal{B}(\mathbb{F}) \cdots \times \mathcal{B}(\mathbb{F})$ is a semialgebra, thus $m_1 = m_2$ due to the measure extension theorem, we write $m_1 = m_2 = \mu^k$.

3. Singular value decomposition on $\mathbb{F}^n$

3.1. **Isometric transformations.** Let $\mathbb{F}$ be a non-Archimedean field, recall that $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ in $\mathbb{R}^n$, we let $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ represent all linear transformations from $\mathbb{F}^n$ to $\mathbb{F}^n$. We choose the base of vector space $\mathbb{F}^n$ as the natural base, then the linear transformations can be regarded as a matrix. Denote $\|T\| = \sup_{\|x\|\neq 0} \frac{\|Tx\|}{\|x\|}$ as the norm of the transformation T.

Proposition 3.1. *For any $T, A, B \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$, we have*
(i) $\|T\| = \max\limits_{1\leq i,j\leq n} \|T_{ij}\|$, *where $T_{i,j}$ represents the element in row i and column j of T.*
(ii) $\|AB\| \leq \|A\| \cdot \|B\|$, $\|A + B\| \leq \max\{\|A\|, \|B\|\}$.

Proof. (i) On the one hand, let T_i be the $i-th$ row vector of T, for any $x \in \mathbb{F}^n$, we have $\|Tx\| \leq \max_i \|T_i x\| \leq (\max_{i,j} \|T_{i,j}\|)(\max_j \|x_j\|) = (\max_{1\leq i,j\leq n} \|T_{ij}\|)\|x\|$. On the other hand, without loss of generality, let $\max_{i,j} \|T_{ij}\| = \|T_{i_0,j_0}\|$, $e_{i_0} = (0, \cdots, 1, \cdots, 0)$, where 1 appears in position i_0. Then

$$\|e_{i_0} \cdot T\| = \|(T_{i_0,1}, T_{i_0,2}, \cdot, T_{i_0,n})\| = \|T_{i_0,j_0}\|.$$

Thus $\|T\| \geq \frac{\|Te_{i_0}\|}{\|e_{i_0}\|} = \|T_{i_0,j_0}\|$.
By using trigonometric inequality and result of (i), we can directly get (ii). □

Remark 3.1. *Let $T \in \mathcal{L}(\mathbb{F}^m, \mathbb{F}^n)$ with $m \neq n$, we still denote*

$$\|T\| = \sup_{\|x\|\neq 0} \frac{\|Tx\|}{\|x\|},$$

where $x \in \mathbb{F}^n$. Under this situation, if A and B are matrices that can be added and multiplied, Proposition 3.1 still holds.

Proposition 3.2. *Let T, U be two $n \times n$ matrices in the filed $\mathbb{F}$. Then*
(i) $\det(TU) = \det T \det U$.
(ii) $\det T \neq 0$ *if and only if its inverse matrix T^{-1} exists. Moreover, $T^{-1} = \frac{T^*}{\det T}$, where T^* denotes the adjoint matrix of T.*
(iii) $\det T \neq 0$ *if and only if the linear equation system $Tx = b$ has a unique solution $x = T^{-1}b$ for any ector b.*
(iv) $\|\det T\| \leq \|T\|^n$.

Proof. The first three statements are similar to the case of $\mathbb{R}^n$, which can be extended to a general field. The last statement is directly from the strong triangle inequality. □

Proposition 3.3. *Let $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$. Then $\|Tx\| = \|x\|$ for any $x \in \mathbb{F}^n$ if and only if $\|T\| = \|\det T\| = 1$.*

Proof. If $\|Tx\| = \|x\|$, for any $x \in \mathbb{F}^n$, then $\|T\| = 1$ and $\|\det T\| \leq 1 = \|T\|^n$. Notice that $Tx = 0$ has only one solution $x = 0$, thus $\det T \neq 0$, and T^{-1} exists. Then for any $x \in \mathbb{F}^n$, we have $\|x\| = \|T \cdot T^{-1}x\| = \|T^{-1}x\|$, so $\|T^{-1}\| = 1$ and $\|\det T^{-1}\| \leq 1$, which imply $\|\det T\| = \|\det T^{-1}\| = 1$. Conversely, we only need to show $\|x\| \leq \|Tx\|$, because $\|Tx\| \leq \|T\| \cdot \|x\| = \|x\|$. By $\|T\| = 1$, we have $\|T^*\| \leq 1$, moreover we get $\|T^{-1}\| = \|\frac{T^*}{\det T}\| = \|T^*\| \leq 1$. Then for any $x \in \mathbb{F}^n$, $\|x\| = \|T^{-1} \cdot Tx\| \leq \|Tx\| \cdot \|T^{-1}\| = \|Tx\|$. □

Remark 3.2. (i) *For convenience, we denote*

$$I_n = \{T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n) : \|T\| = 1 = \|\det T\|\},$$

the set I_n will play the role of the orthogonal transformations set in the case of $\mathbb{R}^n$. We call elements in I_n isometric transformations. For example, the matrix $\begin{pmatrix} 1-c & -c \\ 1 & 1 \end{pmatrix} \in I_2$, *where $c \in F$ with $\|c\| < 1$.*
(ii) *If $T \in I_n$, then $\|\det T\| = 1 > 0$, i.e. T^{-1} exists, thus $\|x\| = \|T \cdot T^{-1}x\| = \|T^{-1}x\|$, for any $x \in \mathbb{F}^n$, then $T^{-1} \in I_n$.*
(iii) *If $A \in I_n$ and $B \in I_n$, then $AB \in I_n$.*
(iv) *For any $P, Q \in I_n$, and any $D \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$, we have $\|PDQ\| = \|D\|$.*

There are three types of elementary isometrics, which correspond to three types of row operations:
(i) Row(column)-multiplying matrices. Multiply all the elements in the i-th row of the identity matrix by a and record that the matrix after multiplication is $T_i(a)$, where $a \in \mathbb{F}$ satisfies $\|a\| = 1$.
(ii) Row(column)-switching matrices. Exchange the positions of rows i and j of the identity matrix, and record the obtained matrix as $T(i,j)$.
(iii) Row(column)-addition matrices. Multiply row i of the identity matrix by a and add it to row j, and record the resulting matrix as $T_{ij}(a)$, where $a \in \mathbb{F}$ with $\|a\| \leq 1$.
We denote the collection of such matrices by $EI_n(\mathbb{F})$(or just EI_n, without causing ambiguity), then, obviously, $EI_n \subset I_n$. More over, $T \in I_n \Leftrightarrow T = T_1 \cdot T_2 \cdots T_k$, $T_i \in EI_n$, $i \leq k$, for some $k \in \mathbb{N}$, this is a direct corollary of the Theorem 3.1.

3.2. **Singular value decomposition.** In the following, we give a singular value decomposition(SVD) of matrix in $\mathbb{F}^n$ and obtain the corresponding singuar value, while Kedlaya [12] gave another direct definition of singular value without introducing SVD.

Theorem 3.1. *Let $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ be a non-singular matrix, then there exists a factorization of the form*

$$T = PDQ,$$

where $P, Q \in I_n$, and $D = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is a diagonal matrix, in which $\sigma_i \in \mathbb{F}$, and $\|\sigma_1\| \geq \|\sigma_2\| \geq \cdots \geq \|\sigma_n\| > 0$. Moreover, the norm of each entry of the diagonal of D is uniquely determined by T.

Proof. Since T is non-singular, there exists $T_{ij} \neq 0$ such that $\|T_{ij}\| = \|T\|$. By repeatedly multiplying T by row(column)-switching matrices, the entry T_{ij} can be moved to the $(1,1)$ position. Hence without loss of generality we assume that $\|T_{11}\| = \|T\| \neq 0$. Then by adding proper multiples of the first row (or column) to the other rows (or columns), T can be converted into a matrix of the form

$$\begin{pmatrix} T_{11} & 0 \\ 0 & * \end{pmatrix}.$$

It is not hard to verify that the above elementary operations can be carried out by left (or right) multiplying T by row(column)-addition matrices. Since T is non-singular, there is at least one non-zero entry in $*$. Repeatedly proceeding this process, the matrix T will ultimately be converted into a diagonal form. Notice that in this process, we only multiply T by elementary isometric matrices. Hence, exist $P, Q \in I_n$ such that T has the form $T = PDQ$, where D is a non-singular diagonal matrix. Moreover, we could label the entries in the diagonal of D in decreasing order of its norm.

Next we prove the uniqueness. Assume $T = P'D'Q'$, where $P', Q' \in I_p(n)$ and $D' = \mathrm{diag}(\sigma'_1, \sigma'_2, \cdots, \sigma'_n)$ is a diagonal matrix, in which $\sigma'_i \in \mathbb{F}$, and $\|\sigma'_1\| \geq \|\sigma'_2\| \geq \cdots \geq \|\sigma'_n\| > 0$. Next we show $\|\sigma_i\| = \|\sigma'_i\|$, for all $1 \leq i \leq n$.

Since $T = PDQ = P'D'Q'$, we get that

$$P^{-1}P'D' = DQQ'^{-1}.$$

For convenience, write $U = P^{-}P'$, $V = QQ'^{-1}$, then $U, V \in I_n$ and $UD' = DV$. Since

$$\det U = \sum_{j_1 j_2 \cdots j_n} \pm U_{1j_1} U_{2j_2} \cdots U_{nj_n},$$

where the summation is taken over all permutations of $(1, 2 \cdots, n)$, we get that

$$1 = \|\det U\| \leq \max_{j_1 j_2 \cdots j_n} \|U_{1j_1} U_{2j_2} \cdots U_{nj_n}\| \leq \|U\|^n = 1,$$

by using the strong triangle inequality. Hence there exists $(j_1, j_2, \cdots, j_n)$, such that $\|U_{1j_1}\| = \|U_{2j_2}\| \cdots = \|U_{nj_n}\| = 1$. Since $UD' = DV$, for each $1 \leq k \leq n$, $(UD')_{kj_k} = (DV)_{kj_k}$. This gives that

$$\|U_{kj_k} \sigma'_{j_k}\| = \|\sigma'_{j_k}\| = \|V_{kj_k} \sigma_k\| \leq \|\sigma_k\|,$$

for any $1 \leq k \leq n$, since $V \in I_n$. Hence,

$$\|\sigma'_{j_k}\| \leq \|\sigma_k\|, \text{ for any } 1 \leq k \leq n. \tag{3.1}$$

But since $P, Q, P', Q' \in I_p(n)$, we have

$$\|\sigma_1 \sigma_2 \cdots \sigma_n\| = \|\sigma'_{j_1} \sigma'_{j_2} \cdots \sigma'_{j_n}\| = \|\det T\|.$$

So none of the inequality in (3.1) can hold strictly. Combining with the assumption

$$\|\sigma'_1\| \geq \|\sigma'_2\| \geq \cdots \geq \|\sigma'_n\| > 0,$$

we get that $\|\sigma_i\| = \|\sigma'_i\|$, for all $1 \leq i \leq n$. □

Remark 3.3. *Let $\alpha_i = \|\sigma_i\|$, $1 \leq i \leq n$. Then $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$ are uniquely determined by T. We call them the* singular values *of T. Obviously, $\alpha_1 = \|T\| = \|D\|$, moreover, if T is contractive and non-singular, then we have $1 > \|T\| = \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$.*

3.3. **Singular value functions.**

Definition 3.1. *Let $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ be a non-singular matrix, and $s \geq 0$, $\lceil s \rceil$ means the minimum integer greater than or equal to s. The singular value function of T is*

$$\phi^s(T) = \begin{cases} \alpha_1\alpha_2\cdots\alpha_{m-1}\alpha_m^{s-m+1} & \text{if } 0 \leq s \leq n,\ m = \lceil s \rceil \\ (\alpha_1\alpha_2\cdots\alpha_n)^{\frac{s}{n}} = \|\det T\|^{\frac{s}{n}} & \text{if } s > n. \end{cases} \tag{3.2}$$

It is clear that as a function of s, $\phi^s(T)$ is continuous and strictly decreasing, when T is contractive. Next theorem gives the submultiplicativity of ϕ^s.

Theorem 3.2. *For $s \geq 0$, ϕ^s is submultiplicative, i.e.,*

$$\phi^s(TU) \leq \phi^s(T)\phi^s(U) \tag{3.3}$$

for any $T, U \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$.

Before proving Theorem 3.2, we need the following two lemmas.

Lemma 3.1. *Given $1 \leq t \leq n$, let T and U be two $n \times n$ matrices whose entries are taken from a field G. Then for each $t \times t$ submatrix $(TU)_{i_1,i_2,\cdots,i_t}^{j_1,j_2,\cdots,j_t}$ of TU,*

$$\det((TU)_{i_1,i_2,\cdots,i_t}^{j_1,j_2,\cdots,j_t}) = \sum_{k_1,k_2\cdots,k_t} \det(T_{i_1,i_2,\cdots,i_t}^{k_1,k_2,\cdots,k_t}) \cdot \det(U_{k_1,k_2,\cdots,k_t}^{j_1,j_2,\cdots,j_t}), \tag{3.4}$$

where the summation is taken over all possible index strings $1 \leq k_1 < k_2 \cdots < k_t \leq n$.

Proof. When $t = 1$, (3.4) is exactly the ordinary matrix multiplication.

Suppose by induction that (3.4) holds for t. We now prove (3.4) for $t + 1$.

For convenience of marking, we assume that the $(t+1) \times (t+1)$ submatrix in the left side of (3.4) is $(TU)_{1,\cdots,t+1}^{1,\cdots,t+1}$. Then by expanding its determinant along the first row, we get

$$\det((TU)_{1,\cdots,t+1}^{1,\cdots,t+1}) = \sum_{j=1}^{t+1} (-1)^{1+j}(TU)_{1,j} \det(TU)_{2,\cdots,t+1}^{1,\cdots,j-1,j+1,\cdots,t+1}.$$

By using the induction assumption, it follows that

$$\begin{aligned}
&\det((TU)_{1,\cdots,t+1}^{1,\cdots,t+1}) \\
= &\sum_{j=1}^{t+1}\left((-1)^{1+j}\sum_{l_1=1}^{n} T_{1l_1}U_{l_1j} \cdot \sum_{l_2,\cdots,l_{t+1}} \det T_{2,\cdots,t+1}^{l_2,\cdots,l_{t+1}} \cdot \det U_{l_2,\cdots,l_{t+1}}^{1,\cdots,j-1,j+1,\cdots,t+1}\right) \\
= &\sum_{j=1}^{t+1}\left((-1)^{1+j} \sum_{\substack{1 \leq l_1 \leq n, \\ l_2,\cdots,l_{t+1}}} T_{1l_1} \det T_{2,\cdots,t+1}^{l_2,\cdots,l_{t+1}} U_{l_1j} \det U_{l_2,\cdots,l_{t+1}}^{1,\cdots,j-1,j+1,\cdots,t+1}\right) \\
= &\sum_{\substack{1 \leq l_1 \leq n, \\ l_2,\cdots,l_{t+1}}} \left(T_{1l_1} \det T_{2,\cdots,t+1}^{l_2,\cdots,l_{t+1}} \sum_{j=1}^{t+1}(-1)^{1+j}U_{l_1j} \det U_{l_2,\cdots,l_{t+1}}^{1,\cdots,j-1,j+1,\cdots,t+1}\right).
\end{aligned}$$

For fixed $1 \leq l_1 \leq n$, $1 \leq l_2 < \cdots < l_{t+1} \leq n$, consider the term

$$\sum_{j=1}^{t+1}(-1)^{1+j}U_{l_1j} \det U_{l_2,\cdots,l_{t+1}}^{1,\cdots,j-1,j+1,\cdots,t+1}.$$

Notice that if $l_1 \in \{l_k : 2 \le k \le t+1\}$, the above term is equal to 0. Otherwise if $l_1 \notin \{l_k : 2 \le k \le t+1\}$, we have

$$\begin{aligned} & \sum_{j=1}^{t+1} (-1)^{1+j} U_{l_1 j} \det U_{l_2,\cdots,l_{t+1}}^{1,\cdots,j-1,j+1,\cdots,t+1} \\ = \quad & (-1)^{1+\widetilde{l_1}} \sum_{j=1}^{t+1} (-1)^{\widetilde{l_1}+j} U_{l_1 j} \det U_{l_2,\cdots,l_{t+1}}^{1,\cdots,j-1,j+1,\cdots,t+1} \\ = \quad & (-1)^{1+\widetilde{l_1}} \det U_{l_1 \vee \{l_2,\cdots,l_{t+1}\}}^{1,\cdots,t+1}, \end{aligned}$$

where $l_1 \vee \{l_2, \cdots, l_{t+1}\}$ denote the index string consisting of $l_1, l_2, \cdots, l_{t+1}$ in increasing order, and $\widetilde{l_1} = \#\{k : l_k \le l_1\}$.

Hence, we have

$$\begin{aligned} & \det((TU)_{1,\cdots,t+1}^{1,\cdots,t+1}) \\ = \quad & \sum_{\substack{l_1 \vee \{l_2,\cdots,l_{t+1}\}, \\ l_1 \notin \{l_2,\cdots,l_{t+1}\}}} (-1)^{1+\widetilde{l_1}} T_{1 l_1} \det T_{2,\cdots,t+1}^{l_2,\cdots,l_{t+1}} \cdot \det U_{l_1 \vee \{l_2,\cdots,l_{t+1}\}}^{1,\cdots,t+1} \\ = \quad & \sum_{k_1,\cdots,k_{t+1}} \left(\det U_{k_1,\cdots,k_{t+1}}^{1,\cdots,t+1} \cdot \sum_{\substack{(l_1 \vee \{l_2,\cdots,l_{t+1}\}) \\ = (k_1,\cdots,k_{t+1})}} (-1)^{1+\widetilde{l_1}} T_{1 l_1} \det T_{2,\cdots,t+1}^{l_2,\cdots,l_{t+1}} \right) \\ = \quad & \sum_{k_1,\cdots,k_{t+1}} \det T_{1,\cdots,t+1}^{k_1,\cdots,k_{t+1}} \cdot \det U_{k_1,\cdots,k_{t+1}}^{1,\cdots,t+1}. \end{aligned}$$

So the lemma is complete. □

Lemma 3.2. *Let $1 \le s \le n$ be an integer. Then, for any non-singular $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$, we have*

$$\phi^s(T) = \max_{T_s} \{\| \det T_s \|\},$$

where the maximum is taken over all $s \times s$ submatrices of T.

Proof. First, there are $P, Q \in I_n$, such that $T = PDQ$ with $D = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is a diagonal matrix, $\sigma_i \in \mathbb{F}$, and $\|\sigma_1\| \ge \|\sigma_2\| \ge \cdots \ge \|\sigma_n\| > 0$.

Let $T_s = T_{i_1,\cdots,i_s}^{j_1,\cdots,j_s}$ be any $s \times s$ submatrix of T. Then by Lemma 3.1, we have

$$\begin{aligned} \det T_s \quad = \quad & \sum_{\substack{k_1,\cdots,k_s, \\ l_1,\cdots,s}} \det P_{i_1,\cdots,i_s}^{k_1,\cdots,k_s} \cdot \det D_{k_1,\cdots,k_s}^{l_1,\cdots,l_s} \cdot \det Q_{l_1,\cdots,l_s}^{j_1,\cdots,j_s} \\ = \quad & \sum_{k_1,\cdots,k_s} \det P_{i_1,\cdots,i_s}^{k_1,\cdots,k_s} \cdot \det D_{k_1,\cdots,k_s}^{k_1,\cdots,k_s} \cdot \det Q_{k_1,\cdots,k_s}^{j_1,\cdots,j_s}. \end{aligned}$$

So by the strong triangle inequality, we have

$$\| \det T_s \| \le \max_{k_1,\cdots,k_s} \| \det P_{i_1,\cdots,i_s}^{k_1,\cdots,k_s} \| \cdot \| \sigma_{k_1} \cdots \sigma_{k_s} \| \cdot \| \det Q_{k_1,\cdots,k_s}^{j_1,\ldots,j_s} \|.$$

Since $P, Q \in I_n$, for each index string $\{k_1, \cdots, k_s\}$, $\| \det P_{i_1,\cdots,i_s}^{k_1,\cdots,k_s} \| \le 1$ and $\| \det Q_{k_1,\cdots,k_s}^{j_1,\ldots,j_s} \| \le 1$. Hence

$$\| \det T_s \| \le \max_{k_1,\cdots,k_s} \| \sigma_{k_1} \cdots \sigma_{k_s} \| \le \alpha_1 \cdots \alpha_s = \phi^s(T).$$

By the arbitrariness of T_s, it follows that $\phi^s(T) \geq \max_{T_s}\{\|\det T_s\|\}$.

Next we show that there must exist a submatrix T_s such that $\phi^s(T) = \|\det T_s\|$. Otherwise, if $\phi^s(T) > \|\det T_s\|$ for every submatrix T_s. By $D = P^{-1}TQ^{-1}$, and $P^{-1}, Q^{-1} \in I_n$, we get

$$\begin{aligned}\sigma_1 \cdots \sigma_s &= \det((P^{-1}TQ^{-1})_{1,\cdots,s}^{1,\cdots,s}) \\ &= \sum_{\substack{k_1,\cdots,k_s,\\ l_1,\cdots,l_s}} \det(P^{-1})_{1,\cdots,s}^{k_1,\cdots,k_s} \det T_{k_1,\cdots,k_s}^{l_1,\cdots,l_s} \det(Q^{-1})_{l_1,\cdots,l_s}^{1,\cdots,s}.\end{aligned}$$

Hence

$$\begin{aligned}\alpha_1 \cdots \alpha_s &\leq \max_{\substack{k_1,\cdots,k_s,\\ l_1,\cdots,l_s}} \|\det(P^{-1})_{1,\cdots,s}^{k_1,\cdots,k_s}\| \cdot \|\det T_{k_1,\cdots,k_s}^{l_1,\cdots,l_s}\| \cdot \|\det(Q^{-1})_{l_1,\cdots,l_s}^{1,\cdots,s}\| \\ &\leq \max_{\substack{k_1,\cdots,k_s,\\ l_1,\cdots,l_s}} \|\det T_{k_1,\cdots,k_s}^{l_1,\cdots,l_s}\| \\ &< \phi^s(T) = \alpha_1 \cdots \alpha_s.\end{aligned}$$

That is, $\alpha_1 \cdots \alpha_s < \alpha_1 \cdots \alpha_s$, which is a contradiction. It follows that $\phi^s(T) = \max_{T_s}\{\|\det T_s\|\}$. □

Proof of Theorem 3.2. If $1 \leq s \leq n$ and s is an integer. Take any $s \times s$ submatrix $(TU)_s$ of TU. Without loss of generality, we may choose $(TU)_s = (TU)_{1,\cdots,s}^{1,\cdots,s}$. Then

$$\det(TU)_{1,\cdots,s}^{1,\cdots,s} = \sum_{k_1,\cdots,k_s} \det T_{1,\cdots,s}^{k_1,\cdots,k_s} \cdot \det U_{k_1,\cdots,k_s}^{1,\cdots,s}.$$

Moreover, we get

$$\|\det(TU)_s\| \leq \max_{k_1,\cdots,k_s} \|\det T_{1,\cdots,s}^{k_1,\cdots,k_s}\| \cdot \|\det U_{k_1,\cdots,k_s}^{1,\cdots,s}\| \leq \phi^s(T)\phi^s(U).$$

Since $(TU)_s$ is an arbitrary $s \times s$ submatrix of TU, we have

$$\max_{(TU)_s} \|\det(TU)_s\| \leq \phi^s(T)\phi^s(U),$$

i.e.

$$\phi^s(TU) \leq \phi^s(T)\phi^s(U).$$

If $1 \leq s \leq n$ and s not an integer, let $m = \lceil s \rceil$, then

$$\begin{aligned}\phi^s(T) = \alpha_1 \cdots \alpha_{m-1}\alpha_m^{s-m+1} &= (\alpha_1 \cdots \alpha_{m-1}\alpha_m)^{s-m+1}(\alpha_1 \cdots \alpha_{m-1})^{m-s} \\ &= (\phi^m(T))^{s-m+1}(\phi^{m-1}(T))^{m-s}.\end{aligned}$$

it follows that

$$\phi^s(TU) = (\phi^m(TU))^{s-m+1}(\phi^{m-1}(TU))^{m-s} \leq \phi^s(T)\phi^s(U).$$

As for $s \geq n$, it is obvious that ϕ^s is multiplicative. □

4. SELF-AFFINE SETS IN $\mathbb{F}^n$

4.1. **Notations.** Firstly, we give some notations. Let $M \geq 2$ be an integer. Denote

$$J_\infty = \{1, 2, \cdots, M\}^{\mathbb{N}},\ J_r = \{(i_1, \cdots, i_r) : 1 \leq i_j \leq M,\ j \leq r\},$$

$J = \bigcup_{r \geq 0} J_r$, and $J_0 = \emptyset$. For any $w \in J$, $v \in J$ or J_∞, denote the length of w by $|w|$, and denote

$$wv$$

to be the sequence obtained by juxtaposition of the terms of w and v. If $v = ww'$, for some $w' \in J_\infty$, we write

$$w < v.$$

If w and $v \in J_\infty$, then

$$w \wedge v$$

is the maximal sequence such that both $w \wedge v < w$ and $w \wedge v < v$. Let

$$d(w, v) = 2^{-|w \wedge v|},$$

for $w \neq v \in J_\infty$. Then (J_∞, d) is a compact metric space. If $w = (i_1, i_2, \cdots, i_k) \in J_k$, we denote

$$T_w = T_{i_1} \cdot T_{i_2} \cdots T_{i_k}.$$

Proposition 4.1. *Let $\{T_1, T_2, \cdots, T_M\}$ be sequence of contractive non-singular linear transformations on $\mathbb{F}^n$.*
(i) *For every two finite words $w, w' \in J$, $\phi^s(T_{ww'}) \leq \phi^s(T_w)\phi^s(T_{w'})$.*
(ii) *There exist $b \leq a < 1$ s.t. $b^{s|w|} \leq \phi^s(T_w) \leq a^{s|w|}$, for any $w \in J$.*

Proof. By using submultiplicativity of ϕ^s, we can get (i). Because T_i is contractive, there exists $b \leq a < 1$, s.t. $1 > a \geq \alpha_1^{(i)} \geq \cdots \geq \alpha_n^{(i)} \geq b > 0$, for any $i \leq M$, where $\alpha_j^{(i)}$ is the j-th singular value of T_i. Then, for any $w \in J$, it is obviously that $\phi^s(T_w) \leq a^{s|w|}$. Notice, $\frac{1}{\alpha_j^{(i)}}$ is the j-th singular value of T_i^{-1}, thus we have

$$\frac{1}{b} \geq \frac{1}{\alpha_n^{(i)}} \geq \cdots \frac{1}{\alpha_1^{(i)}} \geq \frac{1}{a} > 1.$$

It follows that $\phi^s(T_i^{-1}) \leq |\frac{1}{b}|^s$, for any $i \leq M$. Let $w = (w_1, \cdots, w_l)$, then

$$1 = \phi^s(I) \leq \phi^s(T_w)\phi^s(T_{w_1}^{-1}) \cdots \phi^s(T_{w_l}^{-1}) \leq (\frac{1}{b})^{s|w|}\phi^s(T_w).$$

Then we have (ii). □

4.2. **Self-affine set.** Write $\{S_i = T_i + b_i\}_{i \leq M}$ as an *affine iterated function system(AIFS)* on $\mathbb{F}^n$, and let $K(\mathbf{b})$ be the associated invariant set(*self-affine set*), where $\mathbf{b} = (b_1, b_2, \cdots, b_M) \in \mathbb{F}^{nM}$. The existence of $K(\mathbf{b})$ follows from the completeness of the metric space consisting of all compact subsets of $\mathbb{F}^n$ with the Hausdorff metric. Actually, we have classical results about invariant set existence in general metric space, see [6] Theorem 2.5.3.

Lemma 4.1. *Let (X, d) be a complete metric space, $\mathcal{H}(X) = \{E : E$ is compact subset of $X\}$. If d_h is Hausdorff metric in X, then $(\mathcal{H}(X), d_h)$ is complete.*

So the only thing we need to verify is that $S(E) = \bigcup_{i\le M} S_i(E)$ is contraction map on $(\mathcal{H}(\mathbb{F}^n), d_h)$. Actually, for any $A,\ B \in \mathcal{H}(\mathbb{F}^n)$, recall $d_h(A,B) = \max\{d(A,B), d(B,A)\}$, and $d(A,B) = \sup_{x\in A} d(x,B)$, then we have

$$\begin{aligned} d_h(S(A),S(B)) &= d_h(\bigcup_{i\le M} S_i(A), \bigcup_{i\le M} S_i(B)) \\ &\le \max(\max_{1\le i\le M} d(S_i(A),S_i(B)), \max_{1\le i\le M} d(S_i(B),S_i(A))) \\ &= \max(\max_{1\le i\le M} d(T_i(A),T_i(B)), \max_{1\le i\le M} d(T_i(B),T_i(A))) \\ &\le (\max_{1\le i\le M} \|T_i\|)\cdot d_h(A,B). \end{aligned}$$

Thus, by the fixed point theorem, there is a unique $K(\mathbf{b}) \in \mathcal{H}(X)$ such that $S(K(\mathbf{b})) = K(\mathbf{b})$.

Moreover, if $\omega = \omega_1\omega_2\cdots \in J_\infty$, write

$$\begin{aligned} x_\omega(\mathbf{b}) &= \lim_{m\to+\infty}(T_{\omega_1}+b_{\omega_1})(T_{\omega_2}+b_{\omega_2})\cdots(T_{\omega_m}+b_{\omega_m})(0) \\ &= b_{\omega_1} + T_{\omega_1}b_{\omega_2} + T_{\omega_1}T_{\omega_2}b_{\omega_3} + \cdots, \end{aligned}$$

which is well-defined since all the T_i's are contractive. Then, we have $\bigcup_{\omega\in J_\infty} x_\omega(\mathbf{b}) = K(\mathbf{b})$.

4.3. **Hausdorff dimension.** For $s \ge 0$, the s-dimensional Hausdorff measure $\mathcal{H}^s_d$ is defined on subsets of (X,d) by

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A),$$

with

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_i (\mathrm{diam} U_i)^s : A \subset \bigcup_i U_i,\ \mathrm{diam} U_i < \delta\right\},$$

where $\{U_i\}$ are open subsets in X. The *Hausdorff dimension* of $A \subset X$ is

$$\dim_H(A) = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = +\infty\}.$$

See details in [8, 10, 18].

4.4. **net measure on J_∞.** Denote

$$[\omega] = \{\nu \in J_\infty : \omega < \nu\},$$

for any $\omega \in J$. For a finite set $A \subset J$, set A is called the r-cover set of $E(\subset J_\infty)$ if $E \subset \bigcup_{w\in A}[w]$ and $|w| > r$ for any $w \in A$. Let

$$\mathcal{M}^s_r(E) = \inf\{\sum_{w\in A} \phi^s(T_w) :\ A \text{ is r-cover set of } E\},$$

and $\mathcal{M}^s(E) = \lim_{r\to+\infty} \mathcal{M}^s_r(E)$. Then we have following results.

Theorem 4.1. *There exists a unique $s > 0$, s.t. $\lim_{r\to+\infty} \frac{1}{r}\log(\sum_{w\in J_r} \phi^s(T_w)) = 0$, we denote the number s as $d(T_1,\cdots,T_M)$. Moreover, we have the following two equivalent definitions:*
(i) $s = \inf\{s : \mathcal{M}^s(J_\infty) = 0\} = \sup\{s : \mathcal{M}^s(J_\infty) = \infty\}$,
(ii) $s = \inf\{s : \sum_{w\in J} \phi^s(T_w) < \infty\} = \sup\{s : \sum_{w\in J} \phi^s(T_w) = \infty\}$.

Proof. The proof of the Theorem 4.1. is the same as Proposition 4.1 in [9], essentially all that we used was that the $\phi^s(T_i)$ are decreasing in s and its subadditivity, which we already have. Thus, the theorem still holds in $\mathbb{F}$. □

Further, we have the following Lemma, see [9, Lemma 4.2]

Lemma 4.2. *If $\mathcal{M}^s(J_\infty)=\infty$ for some s. Then there exists a compact set $E\subset J_\infty$ such that $0<\mathcal{M}^s(E)<\infty$ and a constant c_1 s.t.*

$$\mathcal{M}^s(E\cap[\omega])\le c_1\phi^s(T_w)\quad(w\in J).$$

5. CALCULATION OF DIMENSION

Proof Theorem 1.1 (i). Let $d(T_1,\cdots,T_M)<s<n$, $s\notin\mathbb{Z}$, and m be the smallest integer greater than or equal to s. Then, by the difintion of $d(T_1,\cdots,T_M)$, we have

$$\lim_{r\to+\infty}\frac{1}{r}\log(\sum_{w\in J_r}\phi^s(T_w))<0,$$

i.e. there exists k s.t.

$$\sum_{w\in J_k}\phi^s(T_w)\le 1.$$

Fix $\epsilon>0$, and for any $w_0=(i_1,i_2,\cdots)\in J_\infty$, let

$$w=(i_1,\cdots,i_{pk})<w_0.$$

We choose $p(p>0)$ to be the smallest number such that $\epsilon\ge\alpha_m>b^k\epsilon$, where α_m is the m-th singular value of T_w and b is defined in Propostion 4.1. Using subadditivity repeatedly, we have

$$\sum_{w\in A}\phi^s(T_w)\le 1,$$

where $A=\{w: w_0\in J_\infty\}$ is a cover set of J_∞, i.e. $J_\infty\subset\bigcup_{w\in A}[w]$. Notice that there exists large enough $R>1$ such that $S_i(B)\subset B$, for any $1\le i\le M$, where $B=\{x\in\mathbb{F}^n:\|x\|\le R\}$. Thus, we have

$$K(\mathbf{b})=\bigcup_{w\in J_\infty}S_w(B)\subset\bigcup_{w\in A}S_w(B).$$

Fix a $w\in A$, by $S_w(B)=T_w(B)+c$, where $c=(c_1,\cdots,c_n)\in\mathbb{F}^n$, we have

$$S_w(B)=\{x\in\mathbb{F}^n:\|x_i-c_i\|\le\alpha_iR,\ 1\le i\le n\},$$

where α_i is the i-singular value of T_w. For any $1\le i<m$, let

$$E_i=\{x\in\mathbb{F}^n:\|x-c_i\|\le\alpha_iR\}=c_i+\sigma_i\cdot y\cdot O,$$

where $\sigma_i,\ y\in\mathbb{F}$, s.t. $\|y\|=R$ and $\|\sigma_i\|=\alpha_i$(this y can be found, otherwise we can make R larger). Notice,

$$O=\sum_{d_1\in R}(d_1+\pi O)=\sum_{d_1,d_2\in R}(d_1+\pi d_2+\pi^2O)=\cdots,$$

where $\sum$ means disjiont union of set and R is a finite set of representatives for O/P with $\mathrm{card}(R)=q=\frac{1}{\|\pi\|}$. Thus

$$E_i=\sum_{d_1\in R}(c_i+\sigma_iyd_1+\sigma_iy\pi O)=\sum_{d_1,d_2\in R}(c_i+\sigma_iyd_1+\sigma_iy\pi d_2+\sigma_iy\pi^2O)=\cdots\ (i<m).$$

That is, E_i can be covered by q^t nonintersecting "balls" of diameter $\frac{\alpha_iR}{q^t}$, i.e. an interval with length $\frac{\alpha_iR}{q^t}$. Then, for any $1\le i<m$, exists $t\in\mathbb{N}$ s.t.

$$q^{t-1}<\frac{\alpha_i}{\alpha_m}\le q^t,$$

then

$$\frac{\alpha_i R}{q^t} \leq \alpha_m R < \frac{\alpha_i R}{q^{t-1}}.$$

So, E_i can be covered by $\frac{\alpha_i q}{\alpha_m}$ intervals of diameter $\alpha_m R$. Moreover, $S_w(B)$ can be covered by

$$\frac{\alpha_1 \cdots \alpha_{m-1} q^{m-1}}{\alpha_m^{m-1}}$$

cubes with side length of $\alpha_m R$. Then we have

$$N_{\alpha_m R}(K(\mathbf{b})) \leq \sum_{w \in A} \phi^s(T_w) \alpha_m^{-s} q^{m-1} \leq b^{-ks} \epsilon^{-s} q^{m-1},$$

where $N_\delta(A)$ is the number of cubes of side δ in a cubical lattice that cover the set A. Thus, we have

$$\overline{\dim}_B K(\mathbf{b}) \leq \overline{\lim}_{\epsilon \to 0} \frac{\log N_{\alpha_m R}(K(\mathbf{b}))}{-\log \alpha_m R} \leq \overline{\lim}_{\epsilon \to 0} (c + s \log \epsilon)/\log \epsilon \leq s,$$

since $\epsilon \geq \alpha_m > b^k \epsilon$. It follows that $\dim_H K(\mathbf{b}) \leq \overline{\dim}_B K(\mathbf{b}) \leq d(T_1, T_2, \cdots, T_M)$ for all $\mathbf{b} \in \mathbb{F}^{nM}$. □

Before proving Theorem 1.1 (ii), we need the following four lemmas.

Lemma 5.1. *Let $r > 0$ and exists $\alpha \in \mathbb{F}$, such that $\|\alpha\| = r$. Then we have the following estimates.*

$$\int_{B_r^n} \|z\|^{-s} d\mu^n(z) = O(r^{n-s}), \text{ as } r \to 0, \text{ for } 0 < s < n,$$

and

$$\int_{\mathbb{F}^n \setminus B_r^n} \|z\|^{-s} d\mu^n(z) = O(r^{n-s}), \text{ as } r \to 0, \text{ for } s > n,$$

where $B_r^n = \{x \in \mathbb{F}^n : \|x\| \leq r\}$.

Proof. Recall, $\pi O = P$, $\|\pi\| = q^{-1}$, then there exists $l \in \mathbb{Z}$ s.t. $q^{-(l+1)} \leq r < q^{-l}$. Let $0 < s < n$, then

$$\begin{aligned}\int_{B_r^n} \|z\|^{-s} d\mu^n(z) &\leq \sum_{i=l}^{+\infty} \int_{B_{q^{-i}}^n \setminus B_{q^{-i-1}}^n} \|z\|^{-s} d\mu^n(z) \\ &= \sum_{i=l}^{+\infty} q^{(i+1)s} (q^{-in} - q^{-(i+1)n}) \\ &= \frac{q^{(s-n)l}}{1 - q^{s-n}} q^s (1 - q^{-n}) \leq C \cdot r^{n-s}.\end{aligned}$$

When $s > n$, similarly, the following formula can be obtained

$$\int_{\mathbb{F}^n \setminus B_r^n} \|z\|^{-s} d\mu^n(z) = O(r^{n-s}).$$

Thus, we got the proof. □

Lemma 5.2. *Let $0 < s < n$ be a non-integral real number. Then there exists a constant c depending only on n, s and r such that*

$$\int_{B_r^n} \frac{d\mu^n(x)}{\|Tx\|^s} \leq \frac{c}{\phi^s(T)}$$

for all non-singular $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$.

Proof. Let $T = PDQ$ be a singular value decomposition of T, where $D = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$, with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$ the associated singular values.

Write

$$I = \int_{B_r^n} \frac{d\mu^n(x)}{\|Tx\|^s},$$

then we have

$$I = \int_{B_r^n} \frac{d\mu^n(x)}{\|DQx\|^s}.$$

Letting $y = Qx$, then $d\mu^n(y) = \|\det Q\| d\mu^n(x) = d\mu^n(x)$ and $\|y\| = \|Qx\| \leq r$, so we obtain

$$I = \int_{B_r^n} \frac{d\mu^n(y)}{\|Dy\|^s} = \int_{B_r^n} \frac{d\mu^n(y)}{\max_{1\leq i\leq n}\{\|\sigma_i y_i\|^s\}}.$$

Let $z_i = \sigma_i y_i d$, where $\|c\| = r^{-1}$, and $c \in \mathbb{F}$, then

$$I = r^n(\alpha_1 \cdots \alpha_n)^{-1} \int_{B_{\alpha_1}\times\cdots\times B_{\alpha_n}} \frac{d\mu(z_1)\cdots d\mu(z_n)}{\max_{1\leq i\leq n}\{\|d\mu(z_i)\|^s\}}.$$

Let $m = \lceil s \rceil$. Writing

$$P_1 = \{z \in B_{\alpha_1} \times \cdots \times B_{\alpha_n} : \max\{\|z_1\|, \cdots, \|z_m\|\} \leq \alpha_m\}$$

and

$$P_2 = \{z \in B_{\alpha_1} \times \cdots \times B_{\alpha_n} : \max\{\|z_1\|, \cdots, \|z_{m-1}\|\} > \alpha_m\},$$

we have $B_{\alpha_1} \times \cdots \times B_{\alpha_n} \subset P_1 \cup P_2$, since $\|z_m\| \leq \alpha_m$. Let

$$I_0 = \int_{B_{\alpha_1}\times\cdots\times B_{\alpha_n}} \frac{d\mu(z_1)\cdots d\mu(z_n)}{\max_{1\leq i\leq n}\{\|z_i\|^s\}}.$$

Then

$$\begin{aligned}
I_0 &\leq \int_{P_1} \frac{d\mu(z_1)\cdots d\mu(z_n)}{\max_{1\leq i\leq m}\{\|z_i\|^s\}} + \int_{P_2} \frac{d\mu(z_1)\cdots d\mu(z_n)}{\max_{1\leq i\leq m-1}\{\|z_i\|^s\}} \\
&\leq \alpha_{m+1}\cdots\alpha_n \int_{\max\{\|z_1\|,\cdots,\|z_m\|\}\leq\alpha_m} \frac{d\mu(z_1)\cdots d\mu(z_m)}{\max_{1\leq i\leq m}\{\|z_i\|^s\}} \\
&\quad +\alpha_m\cdots\alpha_n \int_{\max\{\|z_1\|,\cdots,\|z_{m-1}\|\}>\alpha_m} \frac{d\mu(z_1)\cdots d\mu(z_{m-1})}{\max_{1\leq i\leq m-1}\{\|z_i\|^s\}} \\
&= \alpha_{m+1}\cdots\alpha_n \int_{B_{\alpha_m}^m} \frac{d\mu(z_1)\cdots d\mu(z_m)}{\|z^{(m)}\|^s} + \\
&\quad \alpha_m\cdots\alpha_n \int_{F^{m-1}\setminus B_{\alpha_m}^{m-1}} \frac{d\mu(z_1)\cdots d\mu(z_{m-1})}{\|z^{(m-1)}\|^s}.
\end{aligned}$$

By Lemma 5.1, we get

$$I_0 \leq c_1\alpha_{m+1}\cdots\alpha_n\alpha_m^{(m-s)} + c_2\alpha_m\cdots\alpha_n\alpha_m^{(m-1-s)}$$

for appropriate constants c_1, c_2 depending only on n, s and r. Hence,

$$I \leq (c_1 + c_2) r^n \frac{1}{\alpha_1 \cdots \alpha_{m-1} \alpha_m^{s-m+1}} = \frac{c}{\phi^s(T)}.$$

The proof is finished. □

Lemma 5.3. *If $s \notin \mathbb{Z}$ and $0 < s < n$ and $\|T_i\| < 1$, for any $1 \leq i \leq M$, then there is a number $0 < c < +\infty$ such that*

$$\int_{\boldsymbol{b} \in B_r^{nM}} \frac{d\mu^{nM}(\boldsymbol{b})}{\|x_\omega(\boldsymbol{b}) - x_{\omega'}(\boldsymbol{b})\|^s} \leq \frac{c}{\phi^s(T_{\omega \wedge \omega'})}$$

for all distinct $\omega, \omega' \in J_\infty$.

Proof. This is non-Archimedean case of Lemma 3.1 from Falconer [9]. Write $\omega = (\omega \wedge \omega')\nu$ and $\omega' = (\omega \wedge \omega')\nu'$, where $\nu, \nu' \in J_\infty$. Let $q = |\omega \wedge \omega'|$,

$$\eta = \max_{1 \leq i \leq M} \|T_i\| < 1.$$

Notice, $\nu_1 \neq \nu_1'$, without loss of generality we may assume that $\nu_1 = 1$ and $\nu_1' = 2$. Then

$$\begin{aligned} x_\nu(\mathbf{b}) - x_{\nu'}(\mathbf{b}) &= b_1 - b_2 + (T_{\omega_{q+1}} b_{\omega_{q+2}} + T_{\omega_{q+1}} T_{\omega_{q+2}} b_{\omega_{q+3}} + \cdots) \\ &\quad -(T_{\omega'_{q+1}} b_{\omega'_{q+2}} + T_{\omega'_{q+1}} T_{\omega'_{q+2}} b_{\omega'_{q+3}} + \cdots) \\ &= b_1 - b_2 + E(\mathbf{b}), \end{aligned}$$

where $E \in \mathcal{L}(\mathbb{F}^{nM}, \mathbb{F}^n)$, Falconer's proof goes through if $\|E\| < 1$. Actually, by non-Archimedean properties of norm, we have $\|E\| = \sup \frac{\|E(\mathbf{b})\|}{\|\mathbf{b}\|} \leq \sup \frac{\eta \|\mathbf{b}\|}{\|\mathbf{b}\|} = \eta < 1$. □

Lemma 5.4. *Let h be a Borel measure on J_∞, with $0 < h(J_\infty) < \infty$. If there exists $s < n$ s.t.*

$$\int_{J_\infty} \int_{J_\infty} \int_{\boldsymbol{b} \in B_r^{nM}} \frac{d\mu^{nM}(\boldsymbol{b}) dh(\omega) dh(\omega')}{\|x_\omega(\boldsymbol{b}) - x_{\omega'}(\boldsymbol{b})\|^s} \leq \infty,$$

then, for μ^{nM}-a.e. $\boldsymbol{b} \in B_r^{nM}$, $\dim_H K(\boldsymbol{b}) \geq s$.

Proof. The proof of Lemma 5.4 is basically the same as the proof of the Lemma 5.2 in [9]. So we skip the proof. □

Proof Theorem 1.1 (ii). First, we proof that for μ^{nM}-a.e. $\mathbf{b} \in B_r^{nM}$, $\dim_H K(\mathbf{b}) \geq \min\{n, d(T_1, \cdots, T_M)$. Fix $R > 0$, let $t \notin \mathbb{Z}$ such that $0 < t < \min\{n, d(T_1, \cdots, T_M)\}$, and choose s such that $t < s < \min\{n, d(T_1, \cdots, T_M)$. Then, we have $\mathcal{M}^s(J_\infty) = \infty$, by using Lemma 4.2, there exists a compact set $E \subset J_\infty$ such that $0 < \mathcal{M}^s(E) < \infty$, and exists a constant c_1 such that

$$\mathcal{M}^s(E \cap [\omega]) \leq c_1 \phi^s(T_w) \ \ (w \in J).$$

Define $\nu(A) = \mathcal{M}^s(E \cap A)$, for all $A \in J_\infty$. Then $\nu([w]) \le c_1 \phi^s(T_w)$ for any $w \in J$, and $0 < \nu(J_\infty) = \mathcal{M}^s(E) < \infty$.

$$\begin{aligned}
\int_{J_\infty}\int_{J_\infty}\int_{\mathbf{b}\in B_r^{nM}} \frac{d\mu^{nM}(\mathbf{b})d\nu(\omega)d\nu(\omega')}{\|x_\omega(\mathbf{b}) - x_{\omega'}(\mathbf{b})\|^s} &\le c\int_{J_\infty}\int_{J_\infty} \frac{d\nu(w)d\nu(w')}{\phi^t(T_{w\wedge w'})} (\text{ by Lemma 5.3})\\
(\text{let } \upsilon = w \wedge w') &= c\sum_{\upsilon\in J}\sum_{1\le i\ne j\le M} \frac{\nu([\upsilon,i])\nu([\upsilon,j])}{\phi^t(T_\upsilon)}\\
(\text{by } \nu([\upsilon]) = \sum_{1\le i\le M}\nu([\upsilon,i])) &\le c\sum_{\upsilon\in J}\frac{\nu([\upsilon])^2}{\phi^t(T_\upsilon)}\\
&\le c\cdot c_1\sum_{1\le r}\sum_{\upsilon\in J_r}\frac{\phi^s(T_\upsilon)\nu([\upsilon])}{\phi^t(T_\upsilon)}\\
&\le c\cdot c_1\sum_{1\le r}\sum_{\upsilon\in J_r} a^{r(s-t)}\nu([\upsilon])\\
&= c\cdot c_1\frac{a^{s-t}\nu(J_\infty)}{1-a^{s-t}} < \infty.
\end{aligned}$$

Where a is defined in Proposion 4.1 with $a < 1$. Using Lemma 5.4, we have for μ^{nM}-a.e. $\mathbf{b} \in B_R^{nM}$, we have $\dim_H K(\mathbf{b}) \ge t$. Let $R \to \infty$, we have for μ^{nM}-a.e. $\mathbf{b} \in B_r^{nM}$, $\dim_H K(\mathbf{b}) \ge \min\{n, d(T_1, \cdots, T_M)$.
Then, if $\dim_H K(\mathbf{b}) = \min\{n, d(T_1, \cdots, T_M)\}$, since $\underline{\dim}_B K(\mathbf{b}) \ge \dim_H K(\mathbf{b})$ and $\overline{\dim}_B K(\mathbf{b}) \le d(T_1, T_2, \cdots, T_M)$ for all $\mathbf{b} \in \mathbb{F}^{nM}$ which we proved before. It follows that $\dim_B K(\mathbf{b}) = \min\{n, d(T_1, T_2, \cdots, T_M)\}$ for μ^{nM}-a.e. $\mathbf{b} \in \mathbb{F}^{nM}$. □

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